Resonances of the Laplacian for Riemannian symmetric spaces

Lecture 1

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(Quantum or scattering) Resonances

- The Laplace operator on the unit circle S^1 is $\Delta_{S^1} = -\frac{d^2}{d\theta^2}$.
  It is positive, self-adjoint on $L^2(S^1)$ and with discrete spectrum: eigenvectors of the form $e^{in\theta}$ with eigenvalues n^2, where $n \in \mathbb{Z}$, i.e.
  $$\Delta_{S^1} e^{in\theta} = n^2 e^{in\theta} .$$
- The Laplace operator on the real line $\mathbb{R}$ is $\Delta_{\mathbb{R}} = -\frac{d^2}{dx^2}$.
  It is positive, self-adjoint on $L^2(\mathbb{R})$, with continuous spectrum $[0, +\infty)$ and no eigenvalues.
- The resonances are discrete spectral data, a "replacement of eigenvalues" for differential operators H on noncompact domains X.
  They *might* arise when we replace $L^2(X)$ by a dense subspace on which H is no longer self-adjoint.

  E.g.: $H = \Delta_{\mathbb{R}}$, replace $L^2(\mathbb{R})$ by $C_c^\infty(\mathbb{R})$ (=space of compactly supported smooth functions on $\mathbb{R}$)

The early days of the resonances (cf. E. Harrell [3])

The notion of resonances originated in the 30ies in Quantum Mechanics, for Schrödinger operators.

A Schrödinger operator (or Hamiltonian) is a differential operator

$$H = \Delta + V$$

where: $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the in Laplace operator

V is a potential acting as a multiplication operator.

In 1926 Schrödinger studied the Stark effect, i.e. the shifts caused to hydrogen's emission spectrum by the application of a constant field.

The hydrogen Stark Hamiltonian (in scaled units) on $L^2(\mathbb{R}^3)$:

$$H = \Delta - \frac{1}{|x|} + \kappa x_1$$

where $\kappa \geq 0$ is the electrical field strength and the fields acts in the x_1-direction.

In Schrödinger's model, the energies were the eigenvalues of H and the model was based on eigenfunction expansions.

In 1926, an article in *Nature* by Epstein [2] started as follows:

> *The theory of atomic oscillations recently advanced by Schroedinger is of extraordinary importance since it throws a new light on the problems of atomic structure and, at the same time, offers a convenient practical method for calculating the Heisenberg-Born intensity matrices. It seemed desirable to apply it to as many special cases as possible. A complete theory of the Stark effect in hydrogen was, therefore, developed.*

Despite its influence to modern physics, Schrödinger's analysis contained a mistake: the hydrogen Stark Hamiltonian has no eigenvalues if $\kappa > 0$.

This was first noticed by Oppenheimer [4] in 1928. Oppenheimer did not proved it, but referred to a work of Weyl (where it was not proved either).

The non-existence of eigenvalues for the Stark Hamiltonian was first proved by Titchmarsh [5] in 1951.

Although Schrödinger did not recognize this, the "eigenvalues" playing a role in the Stark effect are resonances. The "eigenfunction expansions" are resonant state expansions.

After Schrödinger and Oppeheimer, several quantum physicists implicitly considered something one could regard as a resonant state, a special non-normalized solution of the Schrödinger equation.

But it took long time for quantum physicists to state the basic questions about quantum resonances:

1. What is the definition of a resonance energy?
2. How to determine if it occurs?
3. How can it be computed?
4. What is a “resonant state,” and how to find it?
5. How can the time-decay of a resonance be quantified?

Rigorous mathematical approaches to resonances were elaborated only in the 1970's and 80's.

Resonances of Schrödinger's operators

Consider the Schrödinger operator (or Hamiltonian)

$$H = \Delta + V$$

where: $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator

V is a potential acting as a multiplication operator.

Under suitable assumptions on V, the operator H extends as a self-adjoint operator on $L^2(\mathbb{R}^n)$ with continuous spectrum $\sigma(H) = [0, +\infty[$.

e. g.: H is self-adjoint if V real valued;
if $\lim_{|x|\to\infty} V(x) = 0$ then the spectrum of H is contained in $[0, +\infty[$

For $u \in \mathbb{C} \setminus [0, +\infty[$, the resolvent of H

$$R_H(u) = (H - u)^{-1}$$

is a bdd operator on $L^2(\mathbb{R}^n)$ depending holomorphically in $u \in \mathbb{C} \setminus [0, +\infty[$, i.e.

$$u \in \mathbb{C} \setminus [0, +\infty[\quad \longmapsto \quad R_H(u) \in \mathcal{B}(L^2(\mathbb{R}^n))$$

is a holomorphic function.

As operator on $L^2(\mathbb{R}^n)$, $R_H(u)$ has no analytic extension across its spectrum.

But: can replace $L^2(\mathbb{R}^n)$ by a smaller dense subspace, like $C_c^\infty(\mathbb{R}^n)$ and consider

$$u \in \mathbb{C} \setminus [0, +\infty[\quad \longmapsto \quad R_H(u) \in \mathrm{Hom}(C_c^\infty(\mathbb{R}^n), C_c^{\infty\prime}(\mathbb{R}^n)) = C_c^{\infty\prime}(\mathbb{R}^n \times \mathbb{R}^n)$$

This map might have some continuation across $[0, +\infty[$.

If the continuation turns out to be meromorphic, then the poles are called the resonances of H.

Problems: Existence, location and counting estimates of the resonances.

- If $V = 0$, i.e. $H = \Delta$ is the free Hamiltonian, these questions can be answered using Fourier analysis.
- If $V \neq 0$, many effective approaches combine the known extension of the free resolvent to properties of V.

The resolvent $R_\Delta(u) = (\Delta - u)^{-1}$ of $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ (i.e. $V = 0$)

can be computed via Fourier analysis.

Fourier transform: $\mathcal{F}f(\lambda) = \widehat{f}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\lambda\cdot x}\, dx \qquad (\lambda \in \mathbb{R}^n)$

Fourier inversion: $\mathcal{F}^{-1}g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(\lambda) e^{i\lambda\cdot x}\, d\lambda \qquad (x \in \mathbb{R}^n)$

Plancherel Theorem: $\|\widehat{f}\|_2 = \|f\|_2 \qquad (f \in L^2(\mathbb{R}^n))$.

$\Delta e^{-i\lambda\cdot x} = \lambda\cdot\lambda e^{-i\lambda\cdot x}$. Hence:

$\mathcal{F}\Delta\mathcal{F}^{-1} = M$ (unitary equivalence of Δ and M)

where $M =$ multiplication operator by $\lambda\cdot\lambda$ on $L^2(\mathbb{R}^n)$, i.e. $Mg(\lambda) = \lambda\cdot\lambda g(\lambda)$.

$\rightsquigarrow$ the spectrum of Δ is $\sigma(\Delta) = [0, +\infty)$.

$\mathcal{F}(\Delta - u)^{-1}\mathcal{F}^{-1} = (M-u)^{-1}$ i.e. $R_\Delta(u) = (\Delta - u)^{-1} = \mathcal{F}^{-1}(M-u)^{-1}\mathcal{F}$

Paley-Wiener theorem: $f \in C_c^\infty(\mathbb{R}^n)$ if and only if $\widehat{f}$ is of exponential type and rapidly decreasing, i.e. $\exists R \geq 0$ such that $\sup_{\lambda\in\mathbb{R}^n} e^{-R|\operatorname{Im}\lambda|}(1+|\lambda|)^N|\widehat{f}(\lambda)| < \infty$ for all $N \in \mathbb{N}$.

Thus: for $u \in \mathbb{C} \setminus [0, +\infty)$, $f \in C_c^\infty(\mathbb{R}^n)$, we have $R_\Delta(u)f \in C^\infty(\mathbb{R}^n)$ and

$$[R_\Delta(u)f](x) \asymp \int_{\mathbb{R}^n} \frac{1}{\lambda\cdot\lambda - u}\, \widehat{f}(\lambda)\, e^{ix\cdot\lambda}\, d\lambda \qquad (x \in \mathbb{R}^n)$$

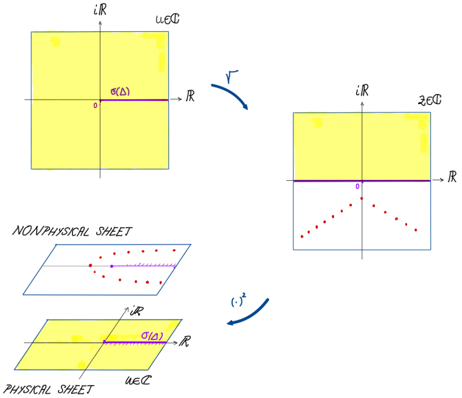
Convenient modifications

- Change variables $u = z^2 \rightsquigarrow$ choice of square root: $\sqrt{-1} = i$
- $u \in \mathbb{C} \setminus [0, +\infty[$ corresponds to $z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$.
- Define

$$R(z) = R_\Delta(z^2) = (\Delta - z^2)^{-1}$$

So $R : \mathbb{C}^+ \to \mathcal{B}(L^2(\mathbb{R}^n))$ is a holomorphic operator-valued function.

Goal: Mero continuation across $\mathbb{R}$ of $R : \mathbb{C}^+ \to \mathrm{Hom}\,(C_c^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$



Meromorphic continuation of the resolvent: the case $n = 1$

Want to continue meromorphically the resolvent of $\Delta = -\dfrac{d^2}{dx^2}$ from $\mathbb{C}^+$ across $\mathbb{R}$:

$$[R(z)f](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\lambda^2 - z^2} \widehat{f}(\lambda)\, e^{ix\lambda}\, d\lambda \qquad (f \in C_c^\infty(\mathbb{R}),\, x \in \mathbb{R},\, \mathrm{Im}\, z > 0)$$

If $\mathrm{Im}\, z > 0$ then $\dfrac{1}{\lambda^2 - z^2}$ is the Fourier transform of $x \mapsto \dfrac{\alpha}{iz} e^{-iz|x|}$, where $\alpha = \sqrt{\frac{2}{\pi}}$.

Since $\mathcal{F}(f * g) = \sqrt{2\pi}(\mathcal{F}f)(\mathcal{F}g)$, the inversion formula for $\mathcal{F}$ yields

$$[R(z)f](x) = \frac{\alpha}{iz}(f * e^{-iz|x|}) = \frac{\alpha}{iz} \int_{\mathbb{R}} e^{iz|x-y|} f(y)\, dy\,.$$

This formula gives a meromorphic extension of $R(z)$ with one pole at $z = 0$.
This pole is the unique resonance of Δ.

The operator $\mathrm{Res}_0 : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$\mathrm{Res}_0 f : x \longmapsto \operatorname*{Res}_{z=0}[R(z)f](x)$$

is the residue operator at the resonance $z = 0$.

The dimension of its image in $C^\infty(\mathbb{R})$ is the rank of the residue operator.

Here: $\operatorname*{Res}_{z=0}[R(z)f](x) = \frac{\alpha}{i} \int_{\mathbb{R}} f(y)\, dy$, constant in x.

The image is $\mathbb{C}$ and the rank is 1.

Meromorphic continuation of the resolvent: the case $n \geq 2$

$$[R(z)f](x) \asymp \int_{\mathbb{R}^n} \frac{1}{\lambda \cdot \lambda - z^2}\, \widehat{f}(\lambda)\, e^{ix\cdot\lambda}\, d\lambda \qquad (f \in C_c^\infty(\mathbb{R}^n),\, x \in \mathbb{R}^n,\, \mathrm{Im}\, z > 0)$$

Wanted: Meromorphic continuation of $R(z)$, for $z \in \mathbb{C}^+$, across $\mathbb{R}$

Idea: polar coordinates

$$[R(z)f](x) \asymp \int_0^{+\infty} \frac{1}{r^2 - z^2} \Big[\underbrace{\underbrace{\Big(\int_{S^{n-1}} e^{ix\cdot rw}\, \widehat{f}(rw)\, dw \Big)}_{\text{even in } r \text{ by } w \mapsto -w} \quad \underbrace{r^{n-2}}_{\text{same parity of } n}}_{\substack{\text{same parity as } n \\ \text{holomorphic in } r \in \mathbb{C} \\ \widehat{f}(rw) \text{ rapidly decreasing by the Paley-Wiener theorem}}} \Big]\, r\, dr$$

$$= \int_0^{+\infty} \frac{1}{r^2 - z^2}\, F(r)\, r\, dr$$

Remark: $F(r) = F(r, f, x)$, but we omit this dependence from the notation.

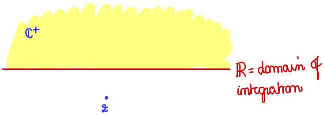
Meromorphic continuation of the resolvent: the case $n > 2$, n odd

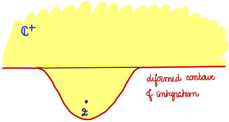
$$[R(z)f](y) \asymp \int_0^{+\infty} \frac{1}{r^2 - z^2} F(r)\, r\, dr$$

Notice:

- $\dfrac{2r}{r^2 - z^2} = \dfrac{1}{r - z} + \dfrac{1}{r + z}$
- $F(r)$ holomorphic in $r \in \mathbb{C}$, rapidly decreasing, with same parity as n. Hence F is odd.

$$\begin{aligned}
[R(z)f](x) &\asymp \int_0^{+\infty} \frac{F(r)}{r - z}\, dr + \int_0^{+\infty} \frac{F(r)}{r + z}\, dr \\
&= \int_0^{+\infty} \frac{F(r)}{r - z}\, dr + \int_{-\infty}^{0} \frac{F(-r)}{-r + z}\, dr \\
&= \int_{-\infty}^{+\infty} \frac{F(r)}{r - z}\, dr
\end{aligned}$$





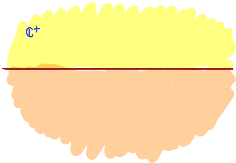
Meromorphic continuation of the resolvent: the case $n \geq 2$, n even

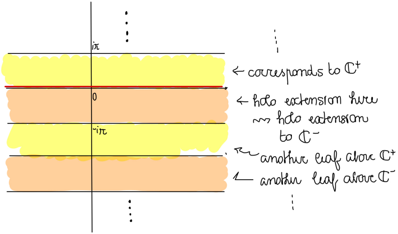
$$[R(z)f](y) \asymp \int_0^{+\infty} \frac{1}{r^2 - z^2}\, F(r)\, r\, dr$$

In this case F is even.

- Change of variables: $r = e^\tau$, $\quad \tau \mapsto F(e^\tau)$ $\;i\pi$-periodic
- $z = e^\zeta \in \mathbb{C}^+ \;\leftrightsquigarrow\; \zeta \in \{0 < \operatorname{Im} w < \pi\}$.

$$[R(e^\zeta)f](y) \asymp \int_{-\infty}^{+\infty} \frac{F(e^\tau)e^{2\tau}}{e^{2\tau} - e^{2\zeta}}\, d\tau$$





Holomorphic extension of the resolvent of the Laplacian on $\mathbb{R}^n$, $n \geq 2$

We have proved:

Theorem

- *For $n \geq 3$ odd, the resolvent R has entire extension to $\mathbb{C}$.*
- *For n even, R extends to be entire as a function of $\log z$, i.e. entire on the logarithmic covering of $\mathbb{C}$.*

Remarks:

- The laplacian Δ has no resonances on $\mathbb{R}^n$ for $n \geq 2$.
- Difference between even and odd dimensional cases.

The case of $H = \Delta + V$ on $\mathbb{R}^n$ when $V \neq 0$

In this case the resonances often exist and play a significant role.

- Suppose $V \in L^\infty_c(\mathbb{R}^n, \mathbb{C})$. Let R_0 denote the resolvent of the free Laplacian. Then

$$(\Delta + V - z^2)R_0(z) = (\Delta - z^2 + V)R_0(z) = I + VR_0(z)$$

We have seen that for $\operatorname{Im} z > 0$

$$(R_0(z)f)^\wedge(\lambda) = \frac{\widehat{f}(\lambda)}{|\lambda|^2 - z^2} \qquad (\lambda \in \mathbb{R}^n)$$

Therefore (by Plancherel)

$$\|R_0(z)\|_{L^2 \to L^2} = \frac{1}{d(z^2, \mathbb{R}^+)} = \frac{1}{|\operatorname{Im}(z^2)|} \leq \frac{1}{|z| \operatorname{Im} z}\,.$$

For $\operatorname{Im} z >> 1$ we have

$$\|VR_0(z)\|_{L^2 \to L^2} \leq \|V\|_\infty (\operatorname{Im} z)^{-2} < 1\,.$$

Hence VR_0 is invertible using the Neumann series

$$(I + VR_0(z))^{-1} = \sum_{k=0}^{\infty} (-1)^k (VR_0(z))^k$$

This shows that

$$R(z) = (\Delta + V - z^2)^{-1} = R_0(z)(I + VR_0(z))^{-1}$$

The study of $R(z)$ reduces to that of $(I + VR_0(z))^{-1}$.

- The meromorphic continuation of $(I + VR_0(z))^{-1}$ relies on the so-called *analytic Fredholm theory*: suppose that $K(z) : L^2 \to L^2$ is a holomorphic family of compact operators for $z \in \mathbb{C}$ and that $(I + K(z_0))^{-1} : L^2 \to L^2$ exists at some $z_0 \in \mathbb{C}$. Then

$$\mathbb{C} \ni z \longmapsto (I + K(z))^{-1} \in B(L^2)$$

  is a meromorphic family of operators.
- As in the free case, the even dimensional case is more complicated.

▷ The proof of the meromorphic continuation of the resolvant $(H - z^2)^{-1}$

▷ the existence and properties of the resonances in this case,

▷ as well as some relvant examples of resonant state expansions,

can be found in the book of Dyalov and Zworski [1]. See also [6].

The theory of resonances of $H = \Delta + V$ appears naturally and plays an important role in many branches of mathematics, physics and engineering.

It is a very active field of research.

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