# Resonances of the Laplacian for Riemannian symmetric spaces

## Lecture 2

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# More general noncompact Riemannian manifolds

The study of analytic/meromorphic continuation of the resolvent of the Laplacian operator (more generally, of Schrödinger operators) has been carried to several classes of complete Riemannian manifolds.

e.g.: hyperbolic or asymptotically hyperbolic manifolds, symmetric or locally symmetric spaces (mostly, of rank 1).

*Many authors:* Borthwick, Bunke, Guillarmou, Guillopé, Mazzeo, Melrose, Müller, Olbrich, Patterson, Perry, Sjöstrand, Strohmaier, Vasy, Zworski,...

*Many motivations/applications:* geometric scattering, dynamical systems, spectral theory, trace formulas...

# Resonances in Geometric Scattering Theory

Let $\Delta$ be (positive) Laplacian on a complete non-compact Riemannian manifold $(X, g)$ (with bounded geometry).

Examples:

- Euclidean space $\mathbb{R}^n$: $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.
- Poincaré half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ with hyperbolic metric: $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$

$\Delta$ is a positive, essentially self-adjoint operator on the Hilbert space $L^2(X)$.
Suppose: $\Delta$ has continuous spectrum $\sigma(\Delta) = [\rho_X^2, +\infty[$ with $\rho_X^2 \geq 0$.

The resolvent of $\Delta$

$$R_\Delta(u) = (\Delta - u)^{-1}$$

is a bdd operator on $L^2(X)$ depending holomorphically on $u \in \mathbb{C} \setminus \sigma(\Delta)$, i.e.

$$\mathbb{C} \setminus \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathcal{B}(L^2(X)).$$

is a holomorphic operator-valued function.

On $L^2(X)$, the resolvent $R_\Delta$ has no extension across $\sigma(\Delta)$.
Letting $R_\Delta$ act on a smaller dense subspace of $L^2(X)$, e.g. $C_c^\infty(X)$, a meromorphic continuation of $R_\Delta$ across $\sigma(\Delta)$ is possible in many cases.

In this case, the poles of the meromorphically extended Laplacian are the resonances of $\Delta$.

# The case of Riemannian symmetric spaces of non-compact type

In the following: $X = G/K$, where:
$G$ = connected noncompact real semisimple Lie group with finite center
$K$ = maximal compact subgroup of $G$

Examples:

- $H^n(\mathbb{R}) = \mathrm{SO}_0(1,n)/\mathrm{SO}(n)$ real hyperbolic space
- $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ upper half-plane
- $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ the space of real positive-definite symmetric $n \times n$ matrices of determinant 1

They are complete non-compact Riemannian manifolds (with bounded geometry) with respect to their canonical $G$-invariant Riemannian metric.

Why studying resonances on $G/K$?

- well understood geometry
- radial part of $\Delta$ on a Cartan subspace is a Schrödinger operator
- Fourier analysis available
- links with Representation Theory (tools and applications)

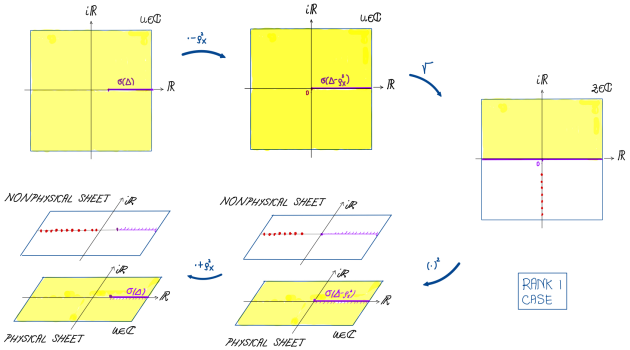
# Convenient modifications

- Consider $\Delta - \rho_X^2$ instead of $\Delta$
  $\rightsquigarrow$ translate the spectrum $[\rho_X^2, +\infty)$ to $[0, +\infty)$
- Change variables $u = z^2$
  $\rightsquigarrow$ $u \in \mathbb{C} \setminus [0, +\infty[$ corresponds to $z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$.
- Define

$$R(z) = R_{\Delta - \rho_X^2}(z^2) = (\Delta - \rho_X^2 - z^2)^{-1}$$

So $R : \mathbb{C}^+ \to \mathcal{B}(L^2(X))$ is a holomorphic operator-valued function.

Goal: Meromorphic continuation across $\mathbb{R}$ of $R : \mathbb{C}^+ \to \mathrm{Hom}\,(C_c^\infty(X), C_c^\infty(X)')$



# Problem 1: Meromorphic continuation and resonances

Wanted: meromorphic continuation of $R : \mathbb{C}^+ \longrightarrow \mathcal{B}(L^2(X))$ across $\mathbb{R}$, by replacing $\mathcal{B}(L^2(X))$ with $\mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$

i.e.

- a Riemann surface $\begin{array}{c} M \\ \downarrow{\scriptstyle \pi} \\ \Omega \end{array}$ with $\Omega$ open in $\mathbb{C}$, containing (a part of) $\mathbb{R}$
- $R : M \to \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$ meromorphic and extending a lift of $R$ to $M$:

$$\begin{array}{ccc} M & \xrightarrow{\;R\;} & \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)') \\ \uparrow & \nearrow_{R} & \\ \Omega \setminus \mathbb{R} & & \end{array}$$

$\forall f, g \in C_c^\infty(X)$: $\langle R(\cdot)f, g\rangle_{L^2(X)}$ lifts and extends to $M$ the function $\langle R(\cdot)f, g\rangle_{L^2(X)}$

If such a meromorphic continuation exists, the poles of the meromorphically extended $R$ will be called the resonances of $\Delta$

(discarding that we are working with $R(z) = R_{\Delta - \rho_X^2}(z^2)$ and not with $R_\Delta(z)$)

**Remark:** as in the case of $\mathbb{R}^n$ we will show (Paley-Wiener theorem) that the image will be in $C^\infty(X)$ and not only in $C_c^\infty(X)'$.

# Residue operators at resonances

Suppose $R : \mathbb{C}^+ \to \mathrm{Hom}(C_c^\infty(X), C^\infty(X))$ extends meromorphically across $\mathbb{R}$.

Let $z_0$ be a resonance (=pole of the extended $R$).

The residue space at $z_0$ is

$$\mathrm{V}_{z_0} = \left\{ \mathrm{Res}_{\varphi(z_0)}(R \circ \varphi^{-1})(f) : f \in C_c^\infty(X) \right\} \subset C^\infty(X)$$

where $\varphi$ is a chart of the Riemann surface in a neighborhood of $z_0$.

The residue operator has finite rank if $\dim \mathrm{V}_{z_0} < \infty$. In this case, $\dim \mathrm{V}_{z_0}$ is its rank.

Since $\Delta$ is $G$-invariant, $\mathrm{V}_{z_0}$ is a $G$-module
(a $K$-spherical representation of $G$).

**Problem 2: identify the resonance representation**

Is the Harish-Chandra module of this representation irreducible?

Is it finite dimensional?

Is it unitarizable?

# The fine structure of $X = G/K$

$X = G/K$ where:
$G$ connected noncompact real semisimple Lie group with finite center
$K$ maximal compact subgroup of $G$

Example: $\mathcal{P}_n = G/K$, $n \geq 2$, where

$$G = \mathrm{SL}(n, \mathbb{R}) = \{g \in \mathrm{Mat}(n, \mathbb{R}) : \det g = 1\}$$
$$K = \mathrm{SO}(n) = \{g \in \mathrm{SL}(n, \mathbb{R}) : gg^t = I\}$$

$\mathcal{P}_n$ is the space of real positive-definite symmetric $n \times n$ matrices of determinant 1

$$G \ni g \curvearrowright \mathcal{P}_n \quad \text{by} \quad X \in \mathcal{P}_n \mapsto gXg^t$$
$$G/K \equiv \mathcal{P}_n \quad \text{by} \quad gK \mapsto gg^t$$

$\mathfrak{g}$= Lie algebra of $G$
$\mathfrak{k}$= Lie algebra of $K$
Then: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
$+1$ & $-1$ eigenspaces of a Cartan involution $\vartheta$.
$\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace
$\dim \mathfrak{a}$=: the (real) rank of $X$

Example: $\mathcal{P}_n$

$$\mathfrak{g} = \{X \in \mathrm{Mat}(n, \mathbb{R}) : \mathrm{Tr}\, X = 0\}, \qquad \vartheta(X) = -X^t$$
$$\mathfrak{k} = \{X \in \mathfrak{g} : X^t = -X\}, \qquad \mathfrak{p} = \{X \in \mathfrak{g} : X^t = X\}$$
$$\mathfrak{a} = \{H = \mathrm{diag}(h_1, h_2, \ldots, h_n) : h_j \in \mathbb{R}, \textstyle\sum_{j=1}^n h_j = 0\} \qquad \rightsquigarrow \mathcal{P}_n \text{ is or rank } n-1.$$

# Root structure of $G/K$

$\mathfrak{a}^*$ = the dual space of $\mathfrak{a}$
$\mathfrak{a}^*_{\mathbb{C}}$ = its complexification
$\Sigma$ == roots of $(\mathfrak{g}, \mathfrak{a})$

$\rightsquigarrow$ $\Sigma$ is a finite subset of $\mathfrak{a}^*$

$\Sigma^+$ = choice of positive positive roots in $\Sigma$

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ = root space of $\alpha \in \Sigma$
$m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ = multiplicity of the root $\alpha$
$\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$
$\langle \cdot, \cdot \rangle$ = restriction to $\mathfrak{a} \times \mathfrak{a}$ of the Killing form of $\mathfrak{g}$

$\rightsquigarrow$ Euclidean structure on $\mathfrak{a}$ and $\mathfrak{a}^*$

$W$ = Weyl group of $\Sigma$
=(finite) subgroup of $\mathrm{O}(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ generated by reflections across $\ker(\alpha)$, $\alpha \in \Sigma^+$

## Examples

$\mathcal{P}_2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$: $\mathfrak{a} = \mathbb{R}H_0$ with $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

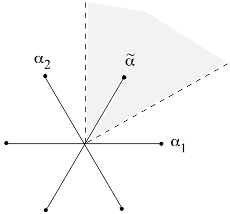
$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then : $[H_0, X] = 2X$, $[H_0, Y] = -2Y$

$\rightsquigarrow \alpha : \mathfrak{a} \to \mathbb{R}$, $\alpha(H) = 2H$ is a root and $\mathfrak{g}_\alpha = \mathbb{R}X$ , $m_\alpha = 1$.

In this case: $\Sigma = \{\pm\alpha\}$, $\Sigma^+ = \{\alpha\}$, $\rho = \frac{1}{2}\alpha$, $W = \{\pm \mathrm{id}\}$

$\mathcal{P}_3 = \mathrm{SL}(3, \mathbb{R}) / \mathrm{SO}(3)$: $\mathfrak{a} = \{H = \mathrm{diag}(h_1, h_2, -(h_1 + h_2)) : h_1, h_2 \in \mathbb{R}\} \cong \mathbb{R}^2$

$\Sigma$ of type $A_2$, $\Sigma^+ = \{\alpha_1, \alpha_2, \widetilde{\alpha} = \alpha_1 + \alpha_2\}$
$m_\alpha = 1$ for all $\alpha$
$W \cong S_3$ generated by reflections across $\ker(\alpha_j)$, $j = 1, 2$




$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$
$N =$ the analytic subgroup of $G$ of Lie algebra $\mathfrak{n}$

Iwasawa decomposition:
$G = KAN = K \exp(\mathfrak{a}) N$
$g = k(g) \exp(H(g)) n(g)$ (uniquely determined components)

# The Helgason-Fourier transform on $X = G/K$

$G = KAN = K\exp(\mathfrak{a})N$ Iwasawa decomposition
$g = k\exp(H(g))\,n$ with $H(g) \in \mathfrak{a}$.

Define:

- $M = Z_K(A)$ and $B = K/M$.
- $\mathcal{A} : X \times B \to \mathfrak{a}$ by $\mathcal{A}(gK, kM) := -H(g^{-1}k)$ ('composite distance')
- For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $b \in B$: $e_{\lambda,b} : X \to \mathbb{C}$ by $e_{\lambda,b}(x) := e^{(\lambda+\rho)\mathcal{A}(x,b)}$

**Helgason-Fourier (HF) transform:**
$\mathcal{F}f(\lambda, b) := \int_X f(x)e_{-\lambda,b}(x)\,dx \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}, b \in B)$

**Plancherel theorem:**
$\mathcal{F}$ extends to a unitary isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^* \times B, |W|^{-1}\,\frac{d\lambda\,db}{c(i\lambda)c(-i\lambda)})$.
where:

$c =$ Harish-Chandra's $c$-function.
$\frac{1}{c(i\lambda)c(-i\lambda)} =$ Plancherel density for the HF transform.

**Paley-Wiener theorem:**

$f \in C_c^\infty(X)$ if and only if $\mathcal{F}f \in H(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$

where

$\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^* \times B)_W = \{F : \mathfrak{a}_{\mathbb{C}}^* \times B \to \mathbb{C}$ entire of uniform exponential type, "$W$-invariant"$\}$

and

$F$ is entire of uniform exponential type if

- $F(\cdot, b)$ entire for all $b \in B$,
- $\exists R \geq 0$ such that $\sup_{(\lambda,b)\in\mathfrak{a}_{\mathbb{C}}^*\times B} e^{-R|\operatorname{Im}\lambda|}(1+|\lambda|)^N|F(\lambda,b)| < \infty$ for all $N \in \mathbb{N}$.

**Helgason-Fourier (HF) inversion, I**

$$\mathcal{F}^{-1}g(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*\times B} g(\lambda,b)e_{i\lambda,b}(x)\ \frac{d\lambda\, db}{c(i\lambda)c(-i\lambda)} \qquad (x \in X)$$

A different expression for the inversion formula holds for $f \in C_c^\infty(X)$.

For $f \in C_c^\infty(X)$

$$\int_B \mathcal{F}f(\lambda, b) e_{i\lambda,b}(x)\, db = (f \times \varphi_{i\lambda})(x)$$

where

$\times$ $=$ convolution on $G/K$

$\rightsquigarrow$ defined by $(f_1 \times f_2)\circ\pi = (f_1\circ\pi)*(f_2\circ\pi)$, where $*$ is the convolution on $G$ and $\pi : G \to G/K$ the canonical projection.

$\varphi_\lambda$ = spherical function on $X$ of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$\rightsquigarrow$ the spherical functions on $X$ are:

- the (normalized) $K$-invariant joint eigenfunctions of the commutative algebra of $G$-invariant diff ops on $X$
- matrix coefficients of the principal $K$-spherical reps of $G$ corresponding to $1_K$

$f \times \varphi_{i\lambda}$ = convolution on $X$ of $f$ and $\varphi_{i\lambda}$
= $[HF$ transform of $f](i\lambda)\varphi_{i\lambda}$, if $f$ right-$K$-invariant

$\rightsquigarrow$ by the Paley-Wiener thm for the HF-transform: entire and rapidly decreasing in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

## Helgason-Fourier (HF) inversion, II

For $g = \mathcal{F}f$ where $f \in C_c^\infty(X)$

$$\mathcal{F}^{-1}g(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} (f \times \varphi_{i\lambda})(x)\, \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \qquad (x \in X)$$

# The resolvent of Laplacian $\Delta$

Explicit formula for the resolvent $R(z)$ of $\Delta$ on $C_c^\infty(X)$ via Fourier transform on $G/K$.

$\Delta e_{-i\lambda,b} = (\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)e_{-i\lambda,b}$
Hence:

$\mathcal{F}\Delta\mathcal{F}^{-1} = M$ (unitary equivalence of $\Delta$ and $M$)
where
$M =$ multiplication operator by $\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle$ on $L^2(\mathfrak{a}^* \times B, |W|^{-1}|c(\lambda)|^{-2}\, d\lambda\, db)$, i.e.
$Mg(\lambda, b) = (\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)g(\lambda, b)$.

$\rightsquigarrow$ the spectrum of $\Delta$ is $\sigma(\Delta) = [\rho_X^2, +\infty)$, where $\rho_X^2 = \langle\rho,\rho\rangle$.

$\mathcal{F}(\Delta - u)^{-1}\mathcal{F}^{-1} = (M-u)^{-1}$ i.e. $R_\Delta(u) = (\Delta - u)^{-1} = \mathcal{F}^{-1}(M-u)^{-1}\mathcal{F}$

Thus: for $z \in \mathbb{C}^+$

$$R(z) = (\Delta - \rho_X^2 - z^2)^{-1} : C_c^\infty(X) \ni f \to R(z)f \in C^\infty(X)$$

is given by

$$[R(z)f](x) \asymp \int_{\mathfrak{a}^*} \frac{1}{\langle\lambda,\lambda\rangle - z^2}\, (f \times \varphi_{i\lambda})(x)\, \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \qquad (y \in X),$$

## Comparison between the cases of $\mathbb{R}^n$ and $X$

The resolvents of the Laplacians of $\mathbb{R}^n$ and $X$ have similar structure:

**Resolvent of the Laplacian on $\mathbb{R}^n$**

$$[R(z)f](x) \asymp \int_{\mathbb{R}^n} \frac{1}{|\lambda|^2 - z^2}\, e^{ix\cdot\lambda}\widehat{f}(\lambda)\, d\lambda \qquad (f \in C_c^\infty(\mathbb{R}^n)\,, x \in \mathbb{R}^n)$$

**Resolvent of the Laplacian on $X = G/K$**

$$[R(z)f](x) \asymp \int_{\mathfrak{a}^*} \frac{1}{\langle\lambda,\lambda\rangle - z^2}\, (f \times \varphi_{i\lambda})(x)\, \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \qquad (f \in C_c^\infty(X)\,, x \in X)$$

where:

$$\begin{aligned} \mathbb{R}^n &\longleftrightarrow \mathfrak{a}^* \\ \text{Euclidean inner product} &\longleftrightarrow \text{inner product induced by Killing form} \\ e^{iy\cdot\lambda}\widehat{f}(\lambda) &\longleftrightarrow (f \times \varphi_{i\lambda})(y) \\ d\lambda &\longleftrightarrow \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \end{aligned}$$

Difference:

In general, the Plancherel density for $X$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$

$\rightsquigarrow$ these singularities *might* originate resonances

Remark: "might" :

- Plancherel density is nonsingular ($\Leftrightarrow$ even multiplicity case): then no resonances
- Plancherel density might be singular, and still no resonances

  *e.g.* $H^n(\mathbb{R}) \times X$ where $n$ odd and $X$ of rank one