Weighted weak type bound for rough maximal singular integrals near L^1

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Joint work with Parasar Mohanty

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Honzík (IMRN, 2020) showed that for $\Omega \in L^\infty(\mathbb{S}^{d-1})$ and $\epsilon > 0$,

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is finite, and $w \in A_\infty$ if

$$[w]_{A_\infty} = \sup_{Q \subset \mathbb{R}^d} \Big(\int_Q w(t)\, dt \Big)^{-1} \Big(\int_Q M(w\chi_Q)(t)\, dt \Big)$$

is finite, where M is the Hardy-Littlewood maximal function.

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Discretising the supremum

Let $\beta \in C_c^\infty(\mathbb{R}^d)$, $supp(\beta) \subset \{\frac{1}{2} \leq |x| \leq 2\}$ and $\sum_{i\in\mathbb{Z}} \beta_i(x) = 1$ for $x \neq 0$, where $\beta_i(x) = \beta(2^{-i}x)$.

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$$T_\Omega^* f \le Mf + \sup_{k\in\mathbb{Z}} \left| \sum_{i>k} K^i * f \right|,$$

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□

It is well known that $\|M\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}$.

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For $\alpha > 0$, we choose a maximal collection $\{Q\}_{Q\in\mathcal{F}}$ of dyadic cubes such that,

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$$f_Q = f\chi_{Q\cap\{|f|>2^{c_1}\alpha\}}.$$

We have $|g| \lesssim \alpha$.

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Further $f_Q = \sum_{n=1}^{\infty} f_Q^n$, where

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Breaking of the Kernel

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $supp(\phi) \subset B(0, \frac{1}{2})$ satisfying $\int \phi = 1$. Set $\phi_j(x) = 2^{jd}\phi(2^j x)$.

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Estimate for $\mathcal{H}_{k,1}$

Fefferman-Stein inequality: For $1 < r < p$, we have

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Expand $(K^i - K_n^i) = \sum_{m=n+1}^{\infty} (K_m^i - K_{m-1}^i)$, $(K_n^i - K_0^i) = \sum_{m=1}^{n} (K_m^i - K_{m-1}^i)$.

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Expand $(K^i - K^i_n) = \sum_{m=n+1}^{\infty} (K^i_m - K^i_{m-1})$, $(K^i_n - K^i_0) = \sum_{m=1}^{n} (K^i_m - K^i_{m-1})$.

We define $T^*_m f = \sup_k \left| \sum_{i>k} (K^i_m - K^i_{m-1}) * f \right|$.

Sparse domination for T^*_m: For $0 < \epsilon < 1$ and $\Delta = [\epsilon^{-1}]$, we have

$$|\langle T^*_m f, g \rangle| \lesssim \frac{2^{-c_2 2^{m-1}}}{\epsilon} \sum_{\mathcal{Q} \in \mathcal{S}} |\mathcal{Q}| \langle f \rangle_{1+\epsilon,\mathcal{Q}} \langle g \rangle_{1+\epsilon,\mathcal{Q}},$$

where $\mathcal{S}$ is a $\frac{1}{2}$-sparse family of cubes.

Fefferman-Stein inequality: Let $1 < r < p$, $(r-1) = 2\epsilon(pr - r + 1)$ and $\Delta = [\epsilon^{-1}]$. Then we have

$$\| T^*_m g \|_{L^p(w)} \lesssim 2^{-c_2 2^{m-1}} p^2 (p')^{\frac{1}{p}} (r')^{1+\frac{1}{p'}} \|g\|_{L^p(M_r w)}.$$

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Breaking of the Kernel

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $supp(\phi) \subset B(0, \frac{1}{2})$ satisfying $\int \phi = 1$. Set $\phi_j(x) = 2^{jd}\phi(2^j x)$. Define

$$K_0^i = K^i * \phi_{-i},$$
$$K_n^i = K^i * \phi_{\Delta 2^n - i} \text{ for } n \in \mathbb{N},$$

where $\Delta \in \mathbb{N}$ depends on the weight w and is to be chosen later. Need to estimate the level set $\{x \in E^c : \sup\limits_{k\in\mathbb{Z}} |\sum\limits_{i>k} K^i * f| > \alpha\}$. We write

$$\begin{aligned}\sum_{i>k} K^i * f = & \sum_{i>k} K^i * g + \sum_{i>k}\sum_{n\geq 1}(K^i - K_n^i) * f^n + \sum_{i>k}\sum_{n\geq 1}(K_n^i - K_0^i) * g^n \\ & + \sum_{i>k}\sum_{n\geq 1} K_0^i * f^n + \sum_{i>k}\sum_{n\geq 1}(K_n^i - K_0^i) * b^n \\ = & \mathcal{H}_{k,1} + \mathcal{H}_{k,2} + \mathcal{H}_{k,3} + \mathcal{H}_{k,CZ} + \mathcal{H}_{k,b}.\end{aligned}$$

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It remains to estimate $w\Big\{x \in E^c : \sup_k \Big|\sum_{i>k}\sum_{n\geq 1}(K_n^i - K_0^i) * b^n\Big| > \frac{\alpha}{5}\Big\}$.

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Finitely many scales s

$$\lesssim \frac{1}{\alpha} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} \sum_{s=3}^{Cns_w - 1} \sum_{Q \in \mathcal{F} : l(Q) = 2^{i-s}} \int \int |(K_n^i - K_0^i)(x-y)||b_Q^n(y)|w(x)$$

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where $s_w=[w]_{A_\infty}\log([w]_{A_\infty}+1)$.

Main Estimate

$$w\Big\{\sup_{k}\Big|\sum_{i>k}\sum_{n=1}^{\infty}\sum_{s=Cns_w}^{\infty}(K_n^i - K_0^i) * B_{i-s}^n\Big| > \frac{\alpha}{10}\Big\}$$

Main Estimate

$$w\Big\{\sum_{s=Cs_w}^{\infty}\sum_{n=1}^{s/Cs_w}\sup_{k}\Big|\sum_{i>k}(K_n^i-K_0^i)*B_{i-s}^n\Big|>\alpha\sum_{s=Cns_w}^{\infty}\lambda_s\Big\}$$

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For $0<\lambda<1$, we have $w(E_\lambda^s)\lesssim\frac{1}{\alpha\lambda^2}2^{-s\delta(1-r_w^{-1})}\|f\|_{L^1(M_{r_w}w)}$.

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Cotlar type inequality:

$\exists$ non-negative functions $\{v_i\}_{i\in\mathbb{Z}}$ with $\sup_{i\in\mathbb{Z}} \|v_i\|_1 \lesssim 1$ such that,

$$\sup_{k\in\mathbb{Z}} \Big|\sum_{i>k} H_m^i * B_{i-s}^n\Big| \lesssim \sum_{r=0}^{\Delta s 2^{n+2}-1} M\Big(\sum_{i\equiv r(mod\ \Delta s 2^{n+2})} H_m^i * B_{i-s}^n\Big) + \Delta s 2^n 2^{-\delta_2 s} \sum_{i\in\mathbb{Z}} \sum_{Q\in\mathcal{F}:l(Q)=2^{i-s}} v_i * |b_Q^n|.$$

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We further break H_m^i as in Seeger (J.AMS, 1996),

$$H_m^i = \Gamma_{m,s}^i + (H_m^i - \Gamma_{m,s}^i).$$

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1. $\text{supp}\,(H_m^i) \subset \{2^{i-2} \leq |x| \leq 2^{i+2}\}$,
2. For each $N \in \mathbb{N}$,

$$\sup_{0\leq l\leq N} \sup_{i\in\mathbb{Z}} r^{d+l}\left|\frac{\partial^l}{\partial r^l} H_m^i(r\theta)\right| \lesssim M_N, \text{ uniformly in } \theta \in \mathbb{S}^{d-1} \text{ and } r \in \mathbb{R}^+.$$

We further break H_m^i as in Seeger (J.AMS, 1996),

$$H_m^i = \Gamma_{m,s}^i + (H_m^i - \Gamma_{m,s}^i).$$

Lemma (Seeger (J.AMS, 1996))

For any index set $I \subset \mathbb{Z}$, we have

$$\left\|\sum_{i\in I} \Gamma_{m,s}^i * B_{i-s}^n\right\|_2^2 \lesssim 2^{-\delta s}\alpha\Big(\sum_{Q\in\mathcal{F}} \|b_Q^n\|_1\Big),$$

Seeger's Decomposition

1. $\mathrm{supp}\,(H^i_m) \subset \{2^{i-2} \leq |x| \leq 2^{i+2}\}$,
2. For each $N \in \mathbb{N}$,

$$\sup_{0\leq l\leq N}\ \sup_{i\in\mathbb{Z}} r^{d+l}\left|\frac{\partial^l}{\partial r^l} H^i_m(r\theta)\right| \lesssim M_N, \text{ uniformly in } \theta \in \mathbb{S}^{d-1} \text{ and } r \in \mathbb{R}^+.$$

We further break H^i_m as in Seeger (J.AMS, 1996),

$$H^i_m = \Gamma^i_{m,s} + (H^i_m - \Gamma^i_{m,s}).$$

Lemma (Seeger (J.AMS, 1996))

For any index set $I \subset \mathbb{Z}$, we have

$$\begin{aligned}\Big\|\sum_{i\in I}\Gamma^i_{m,s} * B^n_{i-s}\Big\|_2^2 &\lesssim 2^{-\delta s}\alpha\Big(\sum_{Q\in\mathcal{F}}\|b^n_Q\|_1\Big),\\ \|(H^i_m - \Gamma^i_{m,s}) * b^n_Q\|_1 &\lesssim 2^{-\delta s}\alpha\|b^n_Q\|_1.\end{aligned}$$

Conclusion

$$
\begin{aligned}
|E_\lambda^s| \lesssim & \Big| \Big\{ \sum_{n=1}^{s/Cs_w} \sum_{m=1}^{n} \Delta s 2^n 2^{-\delta_2 s} \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{F}: l(Q)=2^{i-s}} v_i * |b_Q^n| > \frac{\lambda\alpha}{3} \Big\} \Big| \\
& + \Big| \Big\{ \sum_{n=1}^{s/Cs_w} \sum_{m=1}^{n} \sum_{r=0}^{\Delta s 2^{n+2}-1} M\Big(\sum_{i \equiv r (mod\ \Delta s 2^{n+2})} (\Gamma_{m,s}^i - H_m^i) * B_{i-s}^n \Big) > \frac{\lambda\alpha}{3} \Big\} \\
& + \Big| \Big\{ \sum_{n=1}^{s/Cs_w} \sum_{m=1}^{n} \sum_{r=0}^{\Delta s 2^{n+2}-1} M\Big(\sum_{i \equiv r (mod\ \Delta s 2^{n+2})} \Gamma_{m,s}^i * B_{i-s}^n \Big) > \frac{\lambda\alpha}{3} \Big\} \Big|.
\end{aligned}
$$

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Thank You!