Wavelet system and Muckenhoupt $\mathcal{A}_2$ condition on the Heisenberg group

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Introduction and Motivation

Definition 1.1 (Muckenhoupt $\mathcal{A}_2$ condition)

A non-negative integrable function w on $\mathbb{T}$ is said to satisfy the *Muckenhoupt $\mathcal{A}_2$ condition* if there exists a positive constant C satisfying

$$\left(\frac{1}{|I|}\int_I w(\xi)\,d\xi\right)\left(\frac{1}{|I|}\int_I \frac{1}{w(\xi)}\,d\xi\right) \leq C \tag{1}$$

for all intervals $I \subset \mathbb{T}$.

We recall the following important unitary operators on $L^2(\mathbb{R})$, which generate the system of translates, the Gabor system and the wavelet system on the real line:

Let $\varphi \in L^2(\mathbb{R})$.

- **Translation operator**: For $s \in \mathbb{R}$, $T_s\varphi(x) = \varphi(x - s)$, $x \in \mathbb{R}$.
- **Modulation operator**: For $s \in \mathbb{R}$, $M_s\varphi(x) = e^{2\pi isx}\varphi(x)$, $x \in \mathbb{R}$.
- **Dilation operator**: For $a \in \mathbb{R}^*$, $D_a\varphi(x) = \frac{1}{\sqrt{|a|}}\varphi(\frac{x}{a})$, $x \in \mathbb{R}$.

Brief literature survey

- Nielsen and Šikić[1] - the family $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$, for $\varphi \in L^2(\mathbb{R})$ is a Schauder basis for its closed linear span (shift invariant space generated by φ) if and only if w_φ belongs to the Muckenhoupt $\mathcal{A}_2$ class, where $w_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + k)|^2$, $\xi \in \mathbb{R}$.
- Heil and Powell[2] - the Gabor system $\{M_n T_k\varphi, k, \ n \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{R})$ if and only if $|Z_\varphi|^2 \in \mathcal{A}_2(\mathbb{T} \times \mathbb{T})$, where Z_φ denotes the Zak transform of φ.

[1] M. Nielsen and H. Šikić. "Schauder bases of integer translates". In: *Appl. Comput. Harmon. Anal.* 23(2) (2007), pp. 259–262.

[2] C. Heil and A. M. Powell. "Gabor Schauder bases and the Balian-Low theorem". In: *J. Math. Phys.* 47.11 (2006), p. 113506.

Brief literature survey

- Nielsen[3] - considered the finitely generated shift invariant space $V(\Phi) \subset L^2(\mathbb{R}^d)$, $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ and characterized the system of translates generated by Φ as a Schauder basis in terms of the $\mathcal{A}_2$ condition.
  - In fact,a product Muckenhoupt $\mathcal{A}_2$ condition for matrix weights is used.
  - The weight $W(\Phi) : \mathbb{T}^n \to \mathbb{C}^{N\times N}$ defined by $W(\Phi) = \left(\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}_i(\cdot - k)\overline{\widehat{\varphi}_j(\cdot - k)}\right)_{i,j=1}^{N}$, which is the Gram matrix for Φ is considered.

[3] M. Nielsen. "On stability of finitely generated shift-invariant systems". In: *J. Fourier Anal. Appl.* 16.6 (2010), pp. 901–920.

Aim of the talk

Here, we shall look into a similar problem of characterizing the **wavelet system** on the **Heisenberg group**, arising due to the integer left translations and the nonisotropic dilations, to be a **Schauder basis** for its closed linear span in terms of the **Muckenhoupt** $\mathcal{A}_2$ condition.

Some works on Heisenberg group

- Barbieri et al[4] - shift invariant space on polarized Heisenberg group
- Radha and Saswata[5] - shift invariant space with countably many mutually orthogonal generators on the Heisenberg group
- Das and Radha[6] - shift invariant space with countably many generators

[4] D. Barbieri, E. Hernández, and A. Mayeli. "Bracket map for the Heisenberg group and the characterization of cyclic subspaces". In: *Appl. Comput. Harmon. Anal.* 37(2) (2014), pp. 218–234.

[5] R. Radha and Saswata Adhikari. "Shift-invariant spaces with countably many mutually orthogonal generators on the Heisenberg group". In: *Houston J. Math.* 46.2 (2020), pp. 435–463.

[6] S. R. Das and R. Radha. "Shift-invariant system on the Heisenberg Group". In: *Adv. Oper. Theory* 6.1 (2021), pp. 1–27.

Some works on Heisenberg group

- Mayeli[7] - the existence of a band-limited function $\psi \in L^2(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$ such that the discrete wavelet system $\{L_{2^{-j}\gamma}\delta_{2^{-j}}\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ forms a Parseval frame for $L^2(\mathbb{H})$.
- Arati and Radha[8] - orthonormality of the wavelet system on Heisenberg group
- Radha and Sivananthan[9] - Shannon type sampling theorem for the Heisenberg group

[7] A. Mayeli. "Shannon multiresolution analysis on the Heisenberg group". In: *J. Math. Anal. Appl.* 348(2) (2008), pp. 671 –684.

[8] S. Arati and R. Radha. "Orthonormality of wavelet system on the Heisenberg group". In: *J. Math. Pures Appl.* 131 (2019), pp. 171–192.

[9] R. Radha and S. Sivananthan. "Shannon type sampling theorems on the Heisenberg group". In: *Fields Inst. Comm., Amer. Math. Soc.* 52 (2007), pp. 367–374.

Frames and Riesz bases

Let $\mathcal{H} \neq 0$ be a separable Hilbert space.

Definition 1.2

A sequence $\{f_k : k \in \mathbb{N}\}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k \in \mathbb{N}} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The numbers A and B are called frame bounds. If the right hand side inequality holds, then $\{f_k : k \in \mathbb{N}\}$ is said to be a Bessel sequence with bound B. A frame is called a Parseval frame if $A = B = 1$. A sequence $\{f_k : k \in \mathbb{N}\}$ in $\mathcal{H}$ is said to be a frame sequence if it is a frame for $\overline{\text{span}}\{f_k : k \in \mathbb{N}\}$.

Frames and Riesz bases

Definition 1.3

A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ue_k : k \in \mathbb{N}\}$, where $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded invertible operator. Alternatively, a sequence $\{f_k : k \in \mathbb{N}\}$ is a Riesz basis for $\mathcal{H}$ if $\{f_k : k \in \mathbb{N}\}$ is complete in $\mathcal{H}$, and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}$, one has

$$A \sum_k |c_k|^2 \leq \| \sum_k c_k f_k \|^2 \leq B \sum_k |c_k|^2.$$

A sequence $\{f_k : k \in \mathbb{N}\}$ in $\mathcal{H}$ is called a Riesz sequence if it is a Riesz basis for $\overline{\text{span}}\{f_k : k \in \mathbb{N}\}$.

System of translates (Real line)

Theorem 1.4

Let $\varphi \in L^2(\mathbb{R})$ *and* $w_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + k)|^2$, $\xi \in \mathbb{R}$. *For any* $A, B > 0$, *the following statements*[a] *hold. The collection* $\{T_k\varphi\}_{k\in\mathbb{Z}}$ *is*

(i) *an orthonormal sequence if and only if* $w_\varphi(\xi) = 1$ *a.e.* $\xi \in [0, 1]$.

(ii) *a frame sequence with bounds* A, B *if and only if* $A \leq w_\varphi(\xi) \leq B$ *a.e.* $\xi \in [0, 1] \setminus N$, *where* $N = \{\eta \in [0, 1] : w_\varphi(\eta) = 0\}$.

(iii) *a Riesz sequence with bounds* A, B *if and only if* $A \leq w_\varphi(\xi) \leq B$ *a.e.* $\xi \in [0, 1]$.

[a] O. Christensen. *Frames and bases: An introductory course.* Boston: Birkhäuser, 2008.

System of translates (Heisenberg group)

The following results[10] are on the system of translates $\{L_{(2k,l,m)}\varphi : (k,l,m) \in \mathbb{Z}^{2n+1}\}$ in $L^2(\mathbb{H}^n)$.

Theorem 1.5

If $\varphi \in L^2(\mathbb{H}^n)$, then $\{L_{(2k,l,m)}\varphi : (k,l,m) \in \mathbb{Z}^{2n+1}\}$ is an orthonormal system in $L^2(\mathbb{H}^n)$ if and only if the following conditions hold:

(i) $G^{\varphi}_{0,0}(\lambda) = 1$ *a.e.* $\lambda \in (0,1]$ *and*

(ii) $G^{\varphi}_{k,l}(\lambda) = 0$ *a.e.* $\lambda \in (0,1]$, *for all* $(k,l) \neq (0,0)$ *in* $\mathbb{Z}^{2n}$.

[10] Radha and Adhikari, "Shift-invariant spaces with countably many mutually orthogonal generators on the Heisenberg group".

System of translates (Heisenberg group)

Theorem 1.6

Let $\varphi \in L^2(\mathbb{H}^n)$ *satisfy condition* (ii) *in the above theorem. Then* $\{L_{(2k,l,m)}\varphi : (k,l,m) \in \mathbb{Z}^{2n+1}\}$ *is*

(i) *a frame sequence with bounds* $A, B > 0$ *if and only if*

$$A \leq G^{\varphi}_{0,0}(\lambda) \leq B \quad \text{a.e. } \lambda \in \Omega_{\varphi},$$

where $\Omega_{\varphi} = \{\eta \in (0,1] : G^{\varphi}_{0,0}(\eta) > 0\}$.

(ii) *a Riesz sequence with bounds* $A, B > 0$ *if and only if*

$$A \leq G^{\varphi}_{0,0}(\lambda) \leq B \quad \text{a.e. } \lambda \in (0,1],$$

Heisenberg group

Definition and properties:

- The Heisenberg group $\mathbb{H}^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ satisfying the group law

$$(x, y, t)(u, v, s) = \Big(x + u, y + v, t + s + \tfrac{1}{2}(u \cdot y - v \cdot x)\Big).$$

- It is a nonabelian noncompact locally compact group.
- The Haar measure is the Lebesgue measure $dxdydt$.
- By Stone-von Neumann theorem, every infinite dimensional irreducible unitary representation of the Heisenberg group is unitarily equivalent to the representation π_λ, $\lambda \in \mathbb{R}^*$, where π_λ is defined by

$$\pi_\lambda(x, y, t)\varphi(\xi) = e^{2\pi i\lambda t} e^{2\pi i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y), \quad \varphi \in L^2(\mathbb{R}^n).$$

Group Fourier transform:

- For $f \in L^1(\mathbb{H}^n)$, the group Fourier transform $\hat{f}$ is defined as follows. For $\lambda \in \mathbb{R}^*$, $\hat{f}(\lambda)$ is given by
$$\hat{f}(\lambda) = \int_{\mathbb{C}^n \times \mathbb{R}} f(z,t)\pi_\lambda(z,t)\,dzdt.$$
- $\hat{f}(\lambda)$ is the bounded operator acting on $L^2(\mathbb{R}^n)$ given by
$$\hat{f}(\lambda)\varphi = \int_{\mathbb{C}^n \times \mathbb{R}} f(z,t)\pi_\lambda(z,t)\varphi\,dzdt,\ \varphi \in L^2(\mathbb{R}^n),$$
where the integral is a Bochner integral taking values in the Hilbert space $L^2(\mathbb{R}^n)$.
- $\|\hat{f}(\lambda)\|_{\mathcal{B}} \leq \|f\|_{L^1(\mathbb{H}^n)}$.

Convolution:

- If f and g are in $L^1(\mathbb{H}^n)$, then their convolution is defined by

$$f * g(z,t) = \int\limits_{\mathbb{C}^n \times \mathbb{R}} f((z,t)(w,s)^{-1}) g(w,s)\, dw ds.$$

- Under this convolution, $L^1(\mathbb{H}^n)$ becomes a noncommutative Banach algebra.
- $(\widehat{f * g})(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda), \quad \lambda \in \mathbb{R}^*.$

For $f \in L^2(\mathbb{H}^n)$:

- The definition of the group Fourier transform $\hat{f}$ can also be extended to $L^2(\mathbb{H}^n)$ through the density argument.
- Further, the group Fourier transform satisfies the Plancherel formula
$$\|\hat{f}\|_{L^2(\mathbb{R}^*,\mathcal{B}_2;d\mu)} = \|f\|_{L^2(\mathbb{H}^n)},$$
where, $L^2(\mathbb{R}^*,\mathcal{B}_2;d\mu)$ stands for the space of functions on $\mathbb{R}^*$ taking values in $\mathcal{B}_2$, the class of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$, and square integrable with respect to the Plancherel measure $d\mu(\lambda) = |\lambda|^n d\lambda$.

For a study on the Heisenberg group, we refer to the books by Folland[11] and Thangavelu[12].

[11] G. B. Folland. *Harmonic analysis in phase space*. Princeton, New Jersey: Princeton University Press, 1989.

[12] S. Thangavelu. *Harmonic analysis on the Heisenberg group*. Boston: Birkhäuser, 1998.

Wavelet system on $\mathbb{H}^n$

For $\psi \in L^2(\mathbb{H}^n)$, the left translation and the nonisotropic dilation are defined as

$$L_{(u,v,s)}\psi(x,y,t) = \psi((u,v,s)^{-1}(x,y,t)), \quad (u,v,s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$
$$\text{and } \delta_a\psi(x,y,t) = |a|^{n+1}\psi(ax,ay,a^2t), \quad a \in \mathbb{R}^*,$$

where $(x,y,t) \in \mathbb{H}^n$.

We consider the wavelet system generated by

- integer left translations $L_{(2k,l,m)}$, $(k,l) \in \mathbb{Z}^{2n}$, $m \in \mathbb{Z}$
- nonisotropic dyadic dilations δ_{2^j}, $j \in \mathbb{Z}$,

and denote $\delta_{2^j}L_{(2k,l,m)}\psi$ by $\psi_{j,k,l,m}$ for convenience.

The following was defined by Radha and Saswata[13] while studying the system of translates on the Heisenberg group.

Definition 3.1

For $\psi \in L^2(\mathbb{H}^n)$ and $k, l \in \mathbb{Z}^n$, the function $G^{\psi}_{k,l}$ is defined as

$$G^{\psi}_{k,l}(\lambda) = \sum_{r \in \mathbb{Z}} \langle \widehat{\psi}(\lambda + r), \widehat{L_{(2k,l,0)}\psi}(\lambda + r) \rangle_{\mathcal{B}_2} |\lambda + r|^n, \ \lambda \in (0, 1].$$

[13] Radha and Adhikari, "Shift-invariant spaces with countably many mutually orthogonal generators on the Heisenberg group".

We define the function $H^{\psi}_{j,k,l}$ for $\psi \in L^2(\mathbb{H}^n)$, $j \in \mathbb{Z}$, $k, l \in \mathbb{Z}^n$ as

$$H^{\psi}_{j,k,l}(\lambda) = \sum_{r\in\mathbb{Z}} \langle \widehat{\psi}(2^{2j}(\lambda+r)), (\delta_{2^j} \widehat{L_{(2k,l,0)}\psi})(2^{2j}(\lambda+r))\rangle_{\mathcal{B}_2} |2^{2j}(\lambda+r)|^n,$$

where $\lambda \in (0, 1]$.

Theorem 3.2 (Main result)

Let $\psi \in L^2(\mathbb{H}^n)$ *satisfy*

(i) $G^{\psi}_{k,l}(\lambda) = 0$ *a.e.* $\lambda \in (0, 1]$, *for all* $(k, l) \in \mathbb{Z}^{2n} \setminus \{(0, 0)\}$ *and*

(ii) $H^{\psi}_{j,k,l}(\lambda) = 0$ *a.e.* $\lambda \in (0, 1]$, *for all* $j > 0$ *in* $\mathbb{Z}$ *and* $(k, l) \in \mathbb{Z}^{2n}$,

where $G^{\psi}_{k,l}$ *is as in Definition 3.1. Then, the wavelet system* $\{\psi_{j,k,l,m} : k, l \in \mathbb{Z}^n, j, m \in \mathbb{Z}\}$ *is a Schauder basis for its closed linear span if and only if* $G^{\psi}_{0,0} \in \mathcal{A}_2$.

Notation:

- $\sigma(\mathbb{T})$ - space of trigonometric polynomials on $\mathbb{T}$
- $c_{00}(\mathbb{Z}^{2n+1}, \sigma(\mathbb{T}))$ - space of sequences consisting of only finitely many non-zero terms and each non-zero term is in $\sigma(\mathbb{T})$
- $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ - space of 1-periodic functions on $\mathbb{T}$ taking values in $l^2(\mathbb{Z}^{2n+1})$ and square integrable with respect to $G^{\psi}_{0,0}$
- $\mathcal{A}(\psi)$ - span$\{\psi_{j,k,l,m} : k, l \in \mathbb{Z}^n, j, m \in \mathbb{Z}\}$
- $\mathcal{W}(\psi)$ - $\overline{\text{span}}\{\psi_{j,k,l,m} : k, l \in \mathbb{Z}^n, j, m \in \mathbb{Z}\}$

The proof of the main result makes use of an isometric isomorphism between $\mathcal{W}(\psi)$ and $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

Theorem 3.3

Let $\psi \in L^2(\mathbb{H}^n)$ satisfy (i) and (ii) in Theorem 3.2. For $f \in \mathcal{A}(\psi)$ given by $f = \sum c_{j,k,l,m}\psi_{j,k,l,m}$, the sequence R defined by $R(\lambda) = \{R_{j,k,l}(\lambda)\}_{(j,k,l)\in\mathbb{Z}\times\mathbb{Z}^n\times\mathbb{Z}^n}$, with $R_{j,k,l}(\lambda) = \sum_m c_{j,k,l,m}\, e^{2\pi i m\lambda}$, $\lambda \in \mathbb{T}$ is in $c_{00}(\mathbb{Z}^{2n+1}, \sigma(\mathbb{T}))$. Then the map $f \mapsto R$ defined initially between $\mathcal{A}(\psi)$ and $c_{00}(\mathbb{Z}^{2n+1}, \sigma(\mathbb{T}))$ can be extended to an isometric isomorphism of $\mathcal{W}(\psi)$ onto $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

- Notation used in the proof of the main result:
  - α - the triplet of indices $(j, k, l) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$
  - Λ - $(J, K, L) \in \mathbb{N}^3$
  - Ω - the rectangle $\{(j, k, l) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n : |j| \leq J, |k| \leq K, |l| \leq L\}$
- We assume that $\mathbb{Z}$ is ordered as $\{0, 1, -1, 2, -2, \ldots\}$.

Outline of the proof

For $\alpha \in \mathbb{Z}^{2n+1}$, $m \in \mathbb{Z}$, define

$$R^{\alpha,m}(\lambda) = \{(R^{\alpha,m})_{\alpha'}(\lambda)\}_{\alpha' \in \mathbb{Z}^{2n+1}},$$

$$\text{with } (R^{\alpha,m})_{\alpha'}(\lambda) = \begin{cases} e^{2\pi i m\lambda} & ,\alpha' = \alpha \\ 0 & ,\alpha' \neq \alpha \end{cases}, \text{ where } \lambda \in \mathbb{T}.$$

Then, it can be shown that $\{\psi_{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a Schauder basis for $\mathcal{W}(\psi)$ if and only if $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$, by the isometric isomorphism between these spaces.

Assume that $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a Schauder basis for the space $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

Let $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$. Then there exists a unique $\{c_{\alpha,m}(x)\}_{\alpha\in\mathbb{Z}^{2n+1}, m\in\mathbb{Z}}$ such that $x = \sum_{\alpha,m} c_{\alpha,m}(x)R^{\alpha,m}$. By the Riesz representation theorem, for each $(\alpha, m) \in \mathbb{Z}^{2n+1} \times \mathbb{Z}$, there exists $S^{\alpha,m} \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ such that $c_{\alpha,m}(x) = \langle x, S^{\alpha,m}\rangle$, for every $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

Hence $\langle R^{\alpha',m'}, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} = \delta_{\alpha\alpha'}\delta_{mm'}$, which shows that $\{S^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a **biorthogonal dual** system.

The explicit form of $S^{\alpha,m}$ can be determined and shown to be

$$(S^{\alpha,m})_{\alpha'}(\lambda) = \begin{cases} \frac{1}{G^{\psi}_{0,0}(\lambda)} e^{2\pi i m\lambda} & , \alpha' = \alpha \\ 0 & , \alpha' \neq \alpha \end{cases} \text{ a.e. } \lambda \in \mathbb{T}.$$

Moreover,

$$\begin{aligned} \|S^{\alpha,m}\|^2_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} &= \int_0^1 \|S^{\alpha,m}(\lambda)\|^2_{l^2(\mathbb{Z}^{2n+1})} G^{\psi}_{0,0}(\lambda)\, d\lambda \\ &= \int_0^1 \left| \frac{1}{G^{\psi}_{0,0}(\lambda)} \right| d\lambda, \end{aligned}$$

which shows that $\frac{1}{G^{\psi}_{0,0}} \in L^1(\mathbb{T})$.

We now make use of the result
"A complete sequence $\{x_n : n \in \mathbb{N}\}$ with dual sequence $\{y_n : n \in \mathbb{N}\}$ is a Schauder basis for a Hilbert space $\mathbb{H}$ if and only if the partial sum operators $S_N(x) = \sum_{n=1}^{N} \langle x, y_n \rangle x_n$ are uniformly bounded on $\mathbb{H}$."

So, if we define the partial sum operators $\tilde{T}_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ for $\Lambda \in \mathbb{N}^3$, $M \in \mathbb{N}$ by

$$\tilde{T}_{\Lambda,M}(x) = \sum_{\substack{\alpha \in \Omega \\ |m| \leq M}} \langle x, S^{\alpha,m} \rangle_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} R^{\alpha,m},$$

for $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$, then $\sup_{(\Lambda,M) \in \mathbb{N}^4} \| \tilde{T}_{\Lambda,M} \| < \infty$.

We also consider the symmetric Fourier partial sum operators $T_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ given by

$$T_{\Lambda,M}(x) = \sum_{\substack{\alpha\in\Omega \\ |m|\leq M}} \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} R^{\alpha,m}, \qquad (2)$$

$$\text{for } x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0}).$$

But, we can prove that $\langle x, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} = \langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}$.

So, $\tilde{T}_{\Lambda,M} = T_{\Lambda,M}$ and hence $A := \sup_{(\Lambda,M)\in\mathbb{N}^4} \| T_{\Lambda,M}\| < \infty$.

For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ and $\lambda \in \mathbb{T}$, the terms of the sequence $(T_{\Lambda,M}x)(\lambda)$ can be explicitly computed as

$$(T_{\Lambda,M}x)_{\alpha'}(\lambda) = \begin{cases} \int_0^1 x_{\alpha'}(\lambda')D_M(\lambda-\lambda')d\lambda' & , \alpha' \in \Omega \\ 0 & , \text{otherwise} \end{cases},$$

where D_M is the Dirichlet kernel given by

$$D_M(x) = \sum_{m=-M}^{M} e^{2\pi i m x}.$$

Also, we can choose $N \in \mathbb{N}$, such that for any $M \in \mathbb{N}$,

$D_M(\lambda) \geq \frac{1}{2}\|D_M\|_\infty = \frac{2M+1}{2}$ whenever $|\lambda| \leq \frac{1}{NM}$.

Now, let $I \subseteq \mathbb{T}$ and $|I| > \frac{1}{2N}$. Then

$$\left(\frac{1}{|I|}\int_I G_{0,0}^{\psi}(\lambda)\,d\lambda\right)\left(\frac{1}{|I|}\int_I \frac{1}{G_{0,0}^{\psi}(\lambda)}\,d\lambda\right) \leq (2N)^2\|\psi\|_{L^2(\mathbb{H}^n)}^2 \left\|\frac{1}{G_{0,0}^{\psi}}\right\|_{L^1(\mathbb{T})}.$$

Let $C_1 := (2N)^2\|\psi\|_{L^2(\mathbb{H}^n)}^2\|\frac{1}{G_{0,0}^{\psi}}\|_{L^1(\mathbb{T})}$. Then $0 < C_1 < \infty$ and (1) holds with $C = C_1$.

Let $I \subseteq \mathbb{T}$ and $|I| \leq \frac{1}{2N}$. Choose $M \in \mathbb{N}$ such that $\frac{1}{4NM} \leq |I| \leq \frac{1}{2NM}$.

Define $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ by

$$x(\lambda) = \{x_{\alpha'}(\lambda)\}_{\alpha' \in \mathbb{Z}^{2n+1}}, \text{ with } x_{\alpha'}(\lambda) = \begin{cases} f(\lambda) & , \alpha' = \mathbf{0} \\ 0 & , \text{otherwise} \end{cases},$$

where $f \in L^2(\mathbb{T}; G^{\psi}_{0,0})$, $f \geq 0$ on I and $f = 0$ on $\mathbb{T} \setminus I$. Then for any $\Lambda \in \mathbb{N}^3$,

$$\| T_{\Lambda,M} x \|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} \leq A \|x\|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})}.$$

Consequently, we can prove that

$$\int_I \left| \int_I f(\lambda') D_M(\lambda - \lambda') d\lambda' \right|^2 G^{\psi}_{0,0}(\lambda) d\lambda \leq A^2 \int_I |f(\lambda)|^2 G^{\psi}_{0,0}(\lambda) d\lambda. \tag{3}$$

For $\lambda, \lambda' \in I$, we have $|\lambda - \lambda'| \leq \frac{1}{NM}$ and so $D_M(\lambda - \lambda') \geq \frac{1}{2}(2M+1) \geq \frac{1}{4N|I|}$. Hence

$$\int_I \left| \int_I f(\lambda') D_M(\lambda - \lambda') d\lambda' \right|^2 G^{\psi}_{0,0}(\lambda) d\lambda \geq \frac{1}{(4N|I|)^2} \left(\int_I f(\lambda') d\lambda' \right)^2 \int_I G^{\psi}_{0,0}(\lambda) d\lambda. \quad (4)$$

From (3) and (4), we have

$$\frac{1}{(4N)^2 |I|^2} \left(\int_I f(\lambda') d\lambda' \right)^2 \int_I G^{\psi}_{0,0}(\lambda) d\lambda \leq A^2 \int_I |f(\lambda)|^2 G^{\psi}_{0,0}(\lambda) d\lambda.$$

Let $f = \frac{1}{G^{\psi}_{0,0}}$ on I and $f = 0$ on $\mathbb{T} \setminus I$. Then clearly $f \in L^2(\mathbb{T}; G^{\psi}_{0,0})$. Also,

$$\frac{1}{(4N)^2|I|^2}\left(\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}d\lambda\right)^2\left(\int_I G^{\psi}_{0,0}(\lambda)d\lambda\right) \leq A^2\left(\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}d\lambda\right).$$

In other words,

$$\left(\frac{1}{|I|}\int_I \frac{1}{G^{\psi}_{0,0}(\lambda)}d\lambda\right)\left(\frac{1}{|I|}\int_I G^{\psi}_{0,0}(\lambda)d\lambda\right) \leq A^2(4N)^2.$$

Thus, (1) holds with $C = \max\{C_1, A^2(4N)^2\} > 0$ for all $I \subseteq \mathbb{T}$, thereby proving that $G^{\psi}_{0,0} \in \mathcal{A}_2$.

Conversely, suppose $G^{\psi}_{0,0} \in \mathcal{A}_2$. By using the isometric isomorphism between the spaces $\mathcal{W}(\psi)$ and $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$, we need only show that $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

As $G^{\psi}_{0,0} \in \mathcal{A}_2$, we have $\frac{1}{G^{\psi}_{0,0}} \in L^1(\mathbb{T})$ and so $G^{\psi}_{0,0}(\lambda) > 0$ a.e. $\lambda \in \mathbb{T}$.

For $(\alpha, m) \in \mathbb{Z}^{2n+1} \times \mathbb{Z}$, if we define $S^{\alpha,m} \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ by

$$(S^{\alpha,m})_{\alpha'}(\lambda) = \begin{cases} \frac{1}{G^{\psi}_{0,0}(\lambda)} e^{2\pi i m\lambda} & , \alpha' = \alpha \\ 0 & , \alpha' \neq \alpha \end{cases} \text{ a.e. } \lambda \in \mathbb{T},$$

then $\{S^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a biorthogonal dual to $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$.

Next, we shall show that the operators $T_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ defined in (2) are uniformly bounded.

For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ and $\lambda \in \mathbb{T}$, we shall write

$$(T_{\Lambda,M}x)(\lambda) = \sum_{\alpha\in\Omega} \left(\sum_{|m|\leq M} \langle x, R^{\alpha,m} \rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))} e^{2\pi i m\lambda} \right) \overrightarrow{u^{\alpha}},$$

where $\overrightarrow{u^{\alpha}} = \{(u^{\alpha})_{\alpha'}\}_{\alpha'\in\mathbb{Z}^{2n+1}}$ and $(u^{\alpha})_{\alpha'} = \begin{cases} 1 & , \alpha' = \alpha \\ 0 & , \text{otherwise} \end{cases}$.

Then, it can be further simplified as

$$(T_{\Lambda,M}x)(\lambda) = \sum_{\alpha\in\Omega} \left(\int_0^1 x_{\alpha}(\lambda') D_M(\lambda' - \lambda) d\lambda' \right) \overrightarrow{u^{\alpha}}, \qquad (5)$$

where D_M is the Dirichlet kernel.

In order to find $\|T_{\Lambda,M}x\|$, we also consider the modified partial sum operator $T^*_{\Lambda,M}$ on $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ defined as follows. For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$,

$$T^*_{\Lambda,M}(x) = \sum_{\alpha\in\Omega}\left(\sum_{|m|\leq M-1}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}R^{\alpha,m} + \frac{1}{2}\sum_{|m|=M}\langle x, R^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1}))}R^{\alpha,m}\right).$$

Computing as earlier, we get for $\lambda \in \mathbb{T}$,

$$(T^*_{\Lambda,M}x)(\lambda) = \sum_{\alpha\in\Omega}\left(\int_0^1 x_\alpha(\lambda')D^*_M(\lambda'-\lambda)\,d\lambda'\right)\overrightarrow{u^\alpha}, \tag{6}$$

where D^*_M is the modified Dirichlet kernel[14] given by

$$D^*_M(x) = D_M(x) - \cos 2\pi Mx = \frac{\sin 2\pi Mx}{\tan \pi x}. \tag{7}$$

[14] A. Zygmund. *Trigonometric series, 3rd edition*. Cambridge: Cambridge Univ. Press, 2002.

Defining the sequences $p_M(\lambda)$, $q_M(\lambda)$, $\tilde{p}_M(\lambda)$ and $\tilde{q}_M(\lambda)$[15], for $M \in \mathbb{N}$ and $\lambda \in \mathbb{T}$, as

$$p_M(\lambda) = \{(p_M)_\alpha(\lambda)\}_{\alpha \in \mathbb{Z}^{2n+1}}, \text{ where } (p_M)_\alpha(\lambda) = x_\alpha(\lambda) \cos 2\pi M\lambda,$$
$$q_M(\lambda) = \{(q_M)_\alpha(\lambda)\}_{\alpha \in \mathbb{Z}^{2n+1}}, \text{ where } (q_M)_\alpha(\lambda) = x_\alpha(\lambda) \sin 2\pi M\lambda,$$
$$\tilde{p}_M(\lambda) = \{(\tilde{p}_M)_\alpha(\lambda)\}_{\alpha \in \mathbb{Z}^{2n+1}}, \text{ where } (\tilde{p}_M)_\alpha \text{ is the conjugate function of } (p_M)_\alpha \text{ given by}$$
$$(\tilde{p}_M)_\alpha(\lambda) = -\int_0^1 \frac{(p_M)_\alpha(\lambda + t)}{\tan \pi t}\, dt$$

and $\tilde{q}_M(\lambda) = \{(\tilde{q}_M)_\alpha(\lambda)\}_{\alpha \in \mathbb{Z}^{2n+1}}$, where $(\tilde{q}_M)_\alpha$ is the conjugate function of $(q_M)_\alpha$ given as above,

we have

$$(T^*_{\Lambda,M} x)(\lambda) = \sum_{\alpha \in \Omega} ((\tilde{p}_M)_\alpha(\lambda) \sin 2\pi M\lambda - (\tilde{q}_M)_\alpha(\lambda) \cos 2\pi M\lambda)\overrightarrow{u^\alpha}. \tag{8}$$

[15] Zygmund, *Trigonometric series, 3rd edition*.

For $x \in L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G_{0,0}^{\psi})$, using (5-8), we have

$$
\begin{aligned}
&\| T_{\Lambda,M} x\|_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G_{0,0}^{\psi})} \\
&\qquad \leq \left(\int_0^1 \| T_{\Lambda,M} x(\lambda) - T^*_{\Lambda,M} x(\lambda) \|^2_{l^2(\mathbb{Z}^{2n+1})} G_{0,0}^{\psi}(\lambda)\, d\lambda \right)^{\frac{1}{2}} \\
&\qquad\quad + \left(\int_0^1 \| T^*_{\Lambda,M} x(\lambda) \|^2_{l^2(\mathbb{Z}^{2n+1})} G_{0,0}^{\psi}(\lambda)\, d\lambda \right)^{\frac{1}{2}} \\
&\qquad \leq \left(\int_0^1 \sum_{\alpha \in \Omega} \left(\int_0^1 |x_\alpha(\lambda')|\, d\lambda' \right)^2 G_{0,0}^{\psi}(\lambda)\, d\lambda \right)^{\frac{1}{2}} \\
&\qquad\quad + \left(\int_0^1 \sum_{\alpha \in \Omega} (|(\tilde{p}_M)_\alpha(\lambda)| + |(\tilde{q}_M)_\alpha(\lambda)|)^2 G_{0,0}^{\psi}(\lambda)\, d\lambda \right)^{\frac{1}{2}} \\
&\qquad \leq \sqrt{C} \|x\| + \|\tilde{p}_M\| + \|\tilde{q}_M\|.
\end{aligned}
$$

using Cauchy-Schwarz inequality for the first term.

Using Theorem1[16], there exists $C' > 0$ independent of M, α and x such that

$$\int_0^1 |(\tilde{p}_M)_\alpha(\lambda)|^2 G^{\psi}_{0,0}(\lambda) d\lambda \leq C' \int_0^1 |(p_M)_\alpha(\lambda)|^2 G^{\psi}_{0,0}(\lambda) d\lambda.$$

Then, we get

$$\begin{aligned} \|\tilde{p}_M\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} &\leq C' \sum_{\alpha \in \mathbb{Z}^{2n+1}} \int_0^1 |(p_M)_\alpha(\lambda)|^2 G^{\psi}_{0,0}(\lambda) d\lambda \\ &\leq C' \|x\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})}. \end{aligned}$$

Similarly,

$$\|\tilde{q}_M\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})} \leq C' \|x\|^2_{L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})}.$$

[16] R. Hunt, B. Muckenhoupt, and R. Wheeden. "Weighted norm inequalities for the conjugate function and Hilbert transform". In: *Trans. Math. Soc.* 176 (1973), pp. 227–251.

Thus,

$$\| T_{\Lambda,M}x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} \leq (\sqrt{C}+2\sqrt{C'})\|x\|_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})}.$$

As C and C' are independent of x, Λ and M, we obtain

$$\sup_{(\Lambda,M)\in\mathbb{N}^4} \| T_{\Lambda,M}\| \leq \sqrt{C}+2\sqrt{C'} < \infty.$$

As $\tilde{T}_{\Lambda,M} = T_{\Lambda,M}$, we get $\sup_{(\Lambda,M)\in\mathbb{N}^4} \| \tilde{T}_{\Lambda,M}\| < \infty$.

Let R belong to the space $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$ $= \overline{\text{span}}\{R^{\alpha',m'} : \alpha' \in \mathbb{Z}^{2n+1}, m' \in \mathbb{Z}\}$. By the uniform boundedness of the operators $\tilde{T}_{\Lambda,M}$ and the biorthogonality between $\{R^{\alpha',m'}\}$ and $\{S^{\alpha,m}\}$, we get

$$R = \lim_{\Lambda,M\to\infty} \sum_{\substack{\alpha\in\Omega \\ |m|\leq M}} \langle R, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})} R^{\alpha,m},$$

where the limit is in $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$.

The uniqueness of the coefficients, $\{\langle R, S^{\alpha,m}\rangle_{L^2(\mathbb{T},l^2(\mathbb{Z}^{2n+1});G^{\psi}_{0,0})}\}$ follows again from the biorthogonality between $\{R^{\alpha',m'}\}$ and $\{S^{\alpha,m}\}$.

Thus, $\{R^{\alpha,m} : \alpha \in \mathbb{Z}^{2n+1}, m \in \mathbb{Z}\}$ is a Schauder basis for $L^2(\mathbb{T}, l^2(\mathbb{Z}^{2n+1}); G^{\psi}_{0,0})$, thereby proving our assertion.

Example 3.4

Let $\psi(x, y, t) = \psi_1(x)\psi_2(y)\psi_3(t)$, where $\widehat{\psi_1}$ and ψ_2 are the Haar functions, $\chi^H_{[0,1]}$, on $[0, 1]$, given by

$$\chi^H_{[0,1]}(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $\widehat{\psi_3} = 2\chi_{[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]}$. Then $\psi \in L^2(\mathbb{H})$ satisfies the conditions in the hypothesis of the main result.

The talk is based on the following research paper:

S. Arati and R. Radha, Wavelet system and Muckenhoupt $\mathcal{A}_2$ condition on the Heisenberg group. *Colloquium Mathematicum*, Vol 158, No 1, pp 59-76, DOI: 10.4064/cm7467-9-2018 (2019).

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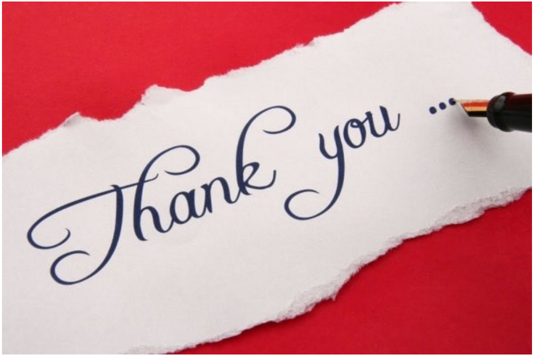
Thank you ...