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On $L^p - L^q$ multiplier

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I will discuss

1) Introduction
2) Preliminary
3) Main Results
4) References

Introduction

For a $f \in \mathcal{S}(\mathbb{C}^n)$ we define its symplectic Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{C}^n} f(x) e^{i \operatorname{Im}(x \cdot \bar{\xi})} dx$$

Then for a tempered distribution T we define its Fourier transform by

$$\hat{T}(\phi) = T(\hat{\phi}), \phi \in \mathcal{S}(\mathbb{C}^n).$$

Given a tempered distribution m we define an operator T by

$$\widehat{Tf} = m\hat{f}, f \in \mathcal{S}(\mathbb{C}^n)$$

If the operator T is $L^p(\mathbb{C}^n)$ to $L^q(\mathbb{C}^n)$ bounded then we say m is an (p,q) multiplier and is denoted by $m \in M_p^q(\mathbb{C}^n)$.

One longstanding problem in Harmonic Analysis is to classify the space $M_p^q(\mathbb{R}^n)$. Only $M_1^1(\mathbb{R}^n)$ and $M_2^2(\mathbb{R}^n)$ is known. They are Fourier transform of finite Borel measure and bounded functions respectively. This is the simple application of Riesz Representation theorem and Plancheral theorem.

Hörmander has proved that

$$M_p^p(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

Also, $M_p^q(\mathbb{R}^n) = M_{q'}^{p'}(\mathbb{R}^n)$.

In particular $M_p^p(\mathbb{R}^n) = M_{p'}^{p'}(\mathbb{R}^n)$.

Another fact is that $M_p^q(\mathbb{R}^n) = \{0\}$ if $p > q$.

One natural question arises that is:

Is $M_p^q(\mathbb{R}^n)$ is interpolation space of the spaces $M_{p_1}^{q_1}(\mathbb{R}^n)$ and $M_{p_2}^{q_2}(\mathbb{R}^n)$ for suitably chosen p and q in terms of p_i & q_i?

Zafran has given the answer and it is not true.

Though the spaces are not classified there is some sufficient conditions available in literature due to Mihlin, Hörmander, Marcinquiwicz, Hahn, Bagby and many others.

Mihlin-Hormander condition is that functions and its derivative upto sufficient order has decay at infinity.

Marcinquiwicz theorem is about if the function is uniformly function of bounded variation on dyadic interval then it is a (p,p)-multiplier for $1 < p < \infty$.

A slightly different approach is given by Hahn. He shows $L^p * L^q \subset M_r^r$ for

$$\frac{1}{p} + \frac{1}{q} = 1 \;\&\; \frac{1}{p} + \frac{1}{r} = 3/2.$$

Later Hahn had improved this result and Bagby gave the full range of it. The result is

If $f \in L^s$, $g \in L^t$ then $f * g \in M_p^q$ where $\left(1 \leq s \leq t, \frac{1}{s}+\frac{1}{t} \geq 1\right)$

i) $1 \leq p \leq q \leq \infty$

ii) $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} + \frac{1}{t} - 1$

iii) $\frac{1}{2} - \frac{1}{t} \leq \frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{2} + \frac{1}{t}$.

So our problem is that for fixed $f \in FL^p(\mathbb{C}^n)$ and $g \in L^q(\mathbb{C}^n)$ can we get new type of multiplier. For that we use twisted convolution. The theorems are

1) $FL^p(\mathbb{C}^n) \times L^{p'}(\mathbb{C}^n) \subset M_1^s(\mathbb{C}^n)$
for $1 \leq p \leq \infty$ and $2 \leq s \leq \infty$.

2) $FL^{p,\infty}(\mathbb{C}^n) \times L^{q,\infty}(\mathbb{C}^n) \subset M_r^s(\mathbb{C}^n)$
where $\frac{1}{s} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 2$.

One is compactly supported.

Preliminary:

The twisted convolution of two functions f & g is defined by

$$f \times g(x) = \int_{\mathbb{C}^n} f(x-y)\, g(y)\, e^{i \operatorname{Im}(x.\bar{y})}\, dy$$

whenever right hand side integral exist.

Two important properties of twisted convolution (not happened for original convolution) are

i) $f, g \in L^1(\mathbb{C}^n)$, $(f \times g)^{\wedge} = \hat{f} \times g$.

ii) $f, g \in L^2(\mathbb{C}^n)$, $\|f \times g\|_2 \leq C \|f\|_2 \|g\|_2$.

Now we state the slight variant of interpolation theorems which is used in the proof our main results.

We define $\quad FL^p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \hat{g},\, g \in L^p(\mathbb{R}^n) \}$

Now we state and prove varient of Riesz-Thorin interpolation theorem.

Lemma: Let T be a linear operator from $FL^{p_1}(\mathbb{R}^n) + FL^{p_2}(\mathbb{R}^n)$ to $L^{q_1}(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n)$ that satisfies

$$\|Tf\|_{q_1} \leq C_1 \|f\|_{FL^{p_1}} \text{ and } \|Tf\|_{q_2} \leq C_2 \|f\|_{FL^{p_2}}$$

then $\quad \|Tf\|_q \leq C \|f\|_{FL^p}$ for $f \in FL^p(\mathbb{R}^n)$ where

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \text{ and } \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad 0 \leq \theta \leq 1.$$

Proof: Given that $\|Tf\|_{q_i} \le C_i \|f\|_{FL^{p_i}}$ for $i=1,2$.

$$\Rightarrow \|Tf\|_{q_i} \le C_i \|\hat{f}\|_{L^{p_i}}$$

$$\Rightarrow \|TF^{-1}(\hat{f})\|_{q_i} \le C_i \|\hat{f}\|_{p_i}$$

$$\Rightarrow \|\tilde{T}(g)\|_{q_i} \le C_i \|g\|_{p_i}$$

where $\tilde{T} = TF^{-1}$ and $g = \hat{f} \in L^{p_i}(\mathbb{R}^n)$.

The operator $\tilde{T}$ is linear. Therefore we apply classical Riesz-Thorin interpolation theorem we get

$$\|\tilde{T}(g)\|_q \le C_1^{\theta} C_2^{1-\theta} \|g\|_p$$

$$\text{i.e. } \|T(f)\|_q \le C_1^{\theta} C_2^{1-\theta} \|f\|_{FL^p}$$

where $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$ & $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$.

Now we state multilinear interpolation theorem.

Lemma: Let T be a multilinear operator is defined on $(FL^{p_1}(\mathbb{R}^n)+FL^{p_2}(\mathbb{R}^n), L^{q_1}(\mathbb{R}^n)+L^{q_2}(\mathbb{R}^n), L^{r_1}(\mathbb{R}^n)+L^{r_2}(\mathbb{R}^n), L^{s_1}(\mathbb{R}^n)+L^{s_2}(\mathbb{R}^n))$ to $L^{t_1}(\mathbb{R}^n)+L^{t_2}(\mathbb{R}^n)$ that satisfies

$$\|T(f_1,f_2,f_3,f_4)\|_{t_i} \le C_i \|f_1\|_{FL^{p_i}(\mathbb{R}^n)} \|f_2\|_{q_i} \|f_3\|_{r_i} \|f_4\|_{s_i}$$

for $i=1,2$.

then $\|T(f_1,f_2,f_3,f_4)\|_t \le C_1^{\theta} C_2^{1-\theta} \|f_1\|_{FL^p} \|f_2\|_{L^q} \|f_3\|_{L^r} \|f_4\|_{L^s}$

where $\frac{1}{t} = \frac{\theta}{t_1} + \frac{1-\theta}{t_2}$, similarly for p, q, r and s.

Now we state Marcinquiwicz interpolation theorem for Lorentz spaces.

Lemma: Let $0<r\leq\infty$, $0<p_0\neq p_1\leq\infty$ and $0<q_0\neq q_1\leq\infty$. Let T be a linear operator defined on $L^{p_0}(\mathbb{R}^n)+L^{p_1}(\mathbb{R}^n)$.

If T satisfies

$$\|T(\chi_A)\|_{L^{q_i,\infty}} \leq M_i\, \mu(A)^{1/p_i}, \quad i=0,1.$$

and let $\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{p_1}$, $\frac{1}{q}=\frac{\theta}{q_0}+\frac{1-\theta}{q_1}$, $0\leq\theta\leq 1$.

Then $\exists$ a constant M such that

$$\|Tf\|_{L^{q,r}} \leq M\,\|f\|_{L^{p,r}}.$$

Main Results

Theorem: If $f \in FL^p(\mathbb{C}^n)$, $g \in L^{p'}(\mathbb{C}^n)$ then $f \times g \in M_1^s(\mathbb{C}^n)$ for $1 \le p \le \infty$ and $2 \le s \le \infty$.

Proof: Define the operator $T : \mathcal{S}(\mathbb{C}^n) \longrightarrow \mathcal{S}'(\mathbb{C}^n)$ by

$$\widehat{Th}(x) = f \times g(x)\, \hat{h}(x).$$

For $2 \le s < \infty$,

$$\|Th\|_s = \sup_{\substack{\phi \in \mathcal{S}(\mathbb{C}^n) \\ \|\phi\|_{s'}=1}} \int_{\mathbb{C}^n} Th(x)\, \phi(x)\, dx$$

$$= \sup_{\substack{\phi \in \mathcal{S}(\mathbb{C}^n) \\ \|\phi\|_{s'}=1}} \int_{\mathbb{C}^n} \widehat{Th}(x)\, \hat{\phi}(x)\, dx$$

$$= \sup_{\substack{\phi \in \mathcal{S}(\mathbb{C}^n) \\ \|\phi\|_{s'}=1}} \int_{\mathbb{C}^n} f \times g(x)\, \hat{h}(x)\, \hat{\phi}(x)\, dx$$

Now, $\left| \int_{\mathbb{C}^n} f \times g(x)\, \hat{h}(x)\, \hat{\phi}(x)\, dx \right| \le \|f\|_2 \|g\|_2 \|h\|_1 \|\phi\|_2$

and $\left| \int_{\mathbb{C}^n} f \times g(x)\, \hat{h}(x)\, \hat{\phi}(x)\, dx \right| \le \|f\|_{FL^{r'}} \|g\|_r \|h\|_1 \|\phi\|_1$, $1 \le r \le \infty$.

Then by multilinear interpolation theorem we get

$$\left| \int_{\mathbb{C}^n} f \times g(x)\, \hat{h}(x)\, \hat{\phi}(x)\, dx \right| \le \|f\|_{FL^p} \|g\|_{p'} \|h\|_1 \|\phi\|_{s'}$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r'}$ & $\frac{1}{s'} = \frac{\theta}{2} + \frac{1-\theta}{1}$

i.e. $\frac{1}{sr'} + \frac{1}{rs'} = \frac{1}{p'}$.

So $f \times g \in M_1^s(\mathbb{C}^n)$ for $2 \leq s < \infty$.

For $s = \infty$,

we define $M_1^\infty(\mathbb{C}^n) = \{\hat{K} : \|K * u\|_\infty \leq C\|u\|_1 , u \in \mathcal{S}(\mathbb{C}^n)\}$

If $K \in L^\infty(\mathbb{C}^n)$ then $\hat{K} \in M_1^\infty(\mathbb{C}^n)$.

For $f \in FL^p(\mathbb{C}^n)$ and $g \in L^{p'}(\mathbb{C}^n)$

$$f \times g \in \widehat{L^\infty}(\mathbb{C}^n)$$

So, $f \times g \in M_1^\infty(\mathbb{C}^n)$.

Now we take the functions from weak L^p spaces and then show its twisted convolution is a multiplier.

Theorem: If $f \in FL^{p,\infty}(\mathbb{C}^n)$ and $g \in L^{q,\infty}(\mathbb{C}^n)$ then

$$f \times g \in M_r^s(\mathbb{C}^n) \text{ for } \frac{1}{s} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 2 .$$

Proof:

$$\|F^{-1}((f \times g)\hat{\phi})\|_s = \|F^{-1}(f \times g) * \phi\|_s$$

$$\leq \|F^{-1}(f \times g)\|_t \|\phi\|_r , \quad \frac{1}{s} + 1 = \frac{1}{t} + \frac{1}{r}$$

$$\leq \|F^{-1}f\|_p \|g\|_q \|\phi\|_r , \quad \frac{1}{t} + 1 = \frac{1}{p} + \frac{1}{q}$$

That is

$$\|F^{-1}((f \times g)\hat{\phi})\|_s \leq \|F^{-1}(f)\|_p \|g\|_q \|\phi\|_r$$

$$\text{where } \frac{1}{s} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 2 .$$

Let, $T(f, g, \Phi] = F^{-1}((f \times g)\hat{\Phi})$.

Then using Marcinquiwicz interpolation theorem for Lorentz space fixing f, Φ, p, r fixed we get

$$\|T(f,g,\Phi)\|_{s,\infty} \leq \|F^{-1}(f)\|_p \|g\|_{q,\infty} \|\Phi\|_r$$

Again using this theorem fixing g, Φ, q, r we get

$$\|T(f,g,\Phi\|_{s,\infty} \leq \|F^{-1}(f)\|_{p,\infty} \|g\|_{q,\infty} \|\Phi\|_{r}$$

We use it for last time we get

$$\|T(f,g,\Phi)\|_{s,r} \leq \|F^{-1}(f)\|_{p,\infty} \|g\|_{q,\infty} \|\Phi\|_{r,r}$$

So, $\|T(f,g,\Phi)\|_s \leq \|F^{-1}(f)\|_{p,\infty} \|g\|_{q,\infty} \|\Phi\|_r$

i.e. $f \times g \in M_r^s(\mathbb{C}^n)$.

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Thank You.