Characterization of Fourier transform of $\mathbb{H}$-valued functions on the Real line

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Outline

1. A brief literature survey
2. Preliminaries
3. The Hilbert C^*-module $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$
4. Fourier transform
5. The main results
6. References

A brief literature survey

- Jaming[ ]- A characterization of Fourier transform [1]

Theorem 1

Let $T : L^1(\mathbb{R}) \to C(\mathbb{R})$ *be a continuous linear operator satisfying* $T(f \star g) = T(f)T(g)$*. Then there exist* $E \subset \mathbb{R}$ *and a function* $\phi : \mathbb{R} \to \mathbb{R}$ *such that* $T(f)(\xi) = \chi_E(\xi)\widehat{f}(\phi(\xi))$, $\xi \in \mathbb{R}$.

[1] P. Jaming, A characterization of Fourier transforms, *Colloq. Math.* **118** (2010), 569-580.

A brief literature survey

- Lakshmi Lavanya and Thangavelu[ ] - A characterisation of the Weyl transform [2]
- Lakshmi Lavanya and Thangavelu[ ] - A characterisation of the Fourier transform on the Heisenberg group [3]
- Shravan and Sivananthan- A characterisation of the Fourier transform on a non-abelian compact group [4]

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[2] R. L. Lavanya and S. Thangavelu, A characterisation of the Weyl transform, *Adv. Pure Appl. Math.* **3** (2012), no. 1, 113-122.

[3] R. L. Lavanya and S. Thangavelu, A characterisation of the Fourier transform on the Heisenberg group, *Ann. Funct. Anal.* **3** (2012), no. 1, 109-120.

[4] N. S. Kumar and S. Sivananthan, Characterisation of the Fourier transform on compact groups, *Bull. Aust. Math. Soc.* **93** (2016), no. 3, 467-472.

Our problem

We wish to ask the same question for H-valued functions on $\mathbb{R}$, where H is a Hilbert C^*-module.

Definition 2

A *C^*-algebra* $(\mathcal{A}, \| \ \|)$ is a Banach algebra equipped with an operator $x \mapsto x^*$(called *involution*) with the following properties.

(i) $(ax + y)^* = \bar{a}x^* + y^*, \ x, y \in \mathcal{A}, a \in \mathbb{C};$

(ii) $(xy)^* = y^*x^*, \ x, y \in \mathcal{A};$

(iii) $(x^*)^* = x, \ x \in \mathcal{A}$; and

(iv) $\|x^*x\| = \|x\|^2, \ x \in \mathcal{A}.$

- If there exists $e \in \mathcal{A}$ such that $ae = ea = a$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is called a *unital* C^*-algebra.

Example 3

(i) The space of all bounded linear operators on a Hilbert space.
(ii) The set of all continuous functions vanishing at infinity on a locally compact Hausdorff space.

Hilbert C^*-module

Hilbert C^*-module was first introduced by Kaplansky [5]. For a study of Hilbert C^*-modules, we refer to[6].

[5] I. Kaplansky, Modules over operator algebras, *Amer. J. Math.*, **75** (1953), 839-858.

[6] E. C. Lance, *Hilbert C^*-modules A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, 1995.

Hilbert C^*-module

Definition 4

Let $\mathcal{A}$ be a C^*-algebra and H be a left $\mathcal{A}$-module (which is also a complex vector space) such that the linear structure on $\mathcal{A}$ and H are compatible, *i.e.* $\lambda(ax) = a(\lambda x) = (\lambda a)x$ for every $\lambda \in \mathbb{C}, a \in \mathcal{A}, x \in \mathrm{H}$. Let $\langle \cdot, \cdot \rangle : \mathrm{H} \times \mathrm{H} \to \mathcal{A}$ be a mapping satisfying

- (i) $\langle x, x \rangle \geq 0$ for every $x \in \mathrm{H}$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
- (ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x, y, z \in \mathrm{H}$,
- (iii) $\langle ax, y \rangle = a\langle x, y \rangle$ for every $a \in \mathcal{A}$ and $x, y \in \mathrm{H}$,
- (iv) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathrm{H}$.

Then the pair $(\mathrm{H}, \langle \cdot, \cdot \rangle)$ is called a *pre Hilbert $\mathcal{A}$-module*. The map $\langle \cdot, \cdot \rangle$ is called an *$\mathcal{A}$-valued inner product*. If H is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ then it is called a *Hilbert $\mathcal{A}$-module*.

A *Hilbert submodule* of a Hilbert module H is a closed submodule of H.

Example 5

(i) Let $\mathcal{A}$ be a C^* algebra. Then $\mathcal{A}$ is a Hilbert C^*-module over itself with respect to the inner product $\langle a, b \rangle = ab^*$.

(ii) Let $\mathrm{H}_k, 1 \leq k \leq n$ be a collection of Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. Then $\mathrm{H} = \oplus_{k=1}^{n} \mathrm{H}_k$ is a Hilbert C^*-module over $\mathcal{A}$ with respect to the inner product $\langle x, y \rangle_{\mathrm{H}} = \sum_{k=1}^{n} \langle x_k, y_k \rangle_{\mathrm{H}_k}$, for $x = (x_1, \cdots x_n), y = (y_1, \cdots y_n) \in \mathrm{H}$.

Remark 6

Let H be a pre Hilbert $\mathcal{A}$-module and $x, y \in \mathrm{H}$. Then one has the Cauchy-Schwarz inequality in H: $\|\langle x, y\rangle\|^2 \leq \|\langle x, x\rangle\| \, \|\langle y, y\rangle\|$.

- Now we shall show that if $(\mathrm{H}, \langle\cdot,\cdot\rangle)$ is a pre Hilbert $\mathcal{A}$-module, then its completion $\overline{\mathrm{H}}$ turns out to be a Hilbert $\mathcal{A}$-module. Clearly, $\overline{\mathrm{H}}$ is a Banach space.
  - For $a \in \mathcal{A}$, $x \in \overline{\mathrm{H}}$, define
$$ax = \lim_{n\to\infty} ax_n,$$
where $\{x_n\}$ is a sequence in H converging to x.
  - It is easy to see that $\overline{\mathrm{H}}$ becomes an $\mathcal{A}$-module with respect to the operation defined above.

Hilbert C^*-module

- For $x, y \in \overline{\mathrm{H}}$ define $\langle \cdot, \cdot \rangle_1$ on $\overline{\mathrm{H}}$ by

$$\langle x, y \rangle_1 = \lim_{n \to \infty} \langle x_n, y_n \rangle,$$

where $\{x_n\}, \{y_n\}$ are two sequences in H converging to x, y respectively.
- It is easy to verify that $(\overline{\mathrm{H}}, \langle \cdot, \cdot \rangle_1)$ is a Hilbert $\mathcal{A}$-module.

Definition 7

Let $(\mathrm{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert C^*-module. A map $T : \mathrm{H} \to \mathrm{H}$ (a priori neither linear nor bounded) is said to be *adjointable* if there exists a map $T^* : \mathrm{H} \to \mathrm{H}$ satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x, y \in \mathrm{H}.$$

An adjointable operator T is said to be *unitary* if $TT^* = T^*T = I$, where I is the identity operator on H.

Hilbert C^*-module

Remark 8

Let $(\mathrm{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert $\mathcal{A}$-module and $T : \mathrm{H} \to \mathrm{H}$ be an adjointable operator. Then T is unitary iff T is onto and preserves the $\mathcal{A}$-valued inner product.

Definition 9

Let $(\mathrm{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module. Then $x \in \mathrm{H}$ is said to be *orthogonal* to $y \in \mathrm{H}$ if $\langle x, y\rangle = 0$ and is denoted by $x \perp y$. If $U \subset \mathrm{H}$ is a Hilbert submodule, then $U^{\perp} = \{x \in \mathrm{H} : \langle x, y\rangle = 0 \textit{ for all } y \in U\}$ is called the *orthogonal complement* of U. The sum $\{u_1 + u_2 : u_i \in U_i, i = 1, 2\}$ of two submodules U_1, U_2 is called *direct* when $U_1 \cap U_2 = \{0\}$ and *orthogonal* when $U_1 \perp U_2$. We shall denote the orthogonal direct sum of U_1 and U_2 by $U_1 \bigoplus U_2$. A Hilbert submodule $U \subset \mathrm{H}$ is said to be *complementable* if $\mathrm{H} = U \bigoplus V$ for some submodule $V \subset \mathrm{H}$.

Hilbert C^*-module

Remark 10

Let H be a Hilbert C^*-module such that $\mathrm{H} = U \bigoplus V$. Then $U^{\perp} = V, \ V^{\perp} = U$.

Remark 11

Let H be a Hilbert C^*-module and $T : \mathrm{H} \to \mathrm{H}$ be an adjointable operator with closed range then $\mathrm{H} = KerT^* \bigoplus Range(T)$. In particular, $Range(T)$ is complementable.

Hilbert C^*-module

Definition 12 ([8])

Let H be a Hilbert C^*-module over a unital C^*-albegra $\mathcal{A}$. A system $\{e_i : i \in I\}$ of vectors from H is said to be orthonormal if $\langle e_i, e_j \rangle = \delta_{i,j} e$, where e is the identity element of $\mathcal{A}$. A orthonormal system is said to be a orthonormal basis for H if for every $x \in \mathrm{H}$ there exist $a_i \in \mathcal{A}$ such that $x = \sum_{i \in I} a_i e_i$.

Example 13

Let $\mathcal{A}$ be a unital C^*-algebra. Consider the Hilbert C^*-module $\mathrm{H} = \underbrace{\mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{n-\text{times}}$. Then $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis for H, where $e_i = (0, 0, \cdots, \underset{\substack{\uparrow \\ i-\text{th}}}{e}, \cdots, 0)$ and e is the identity element of $\mathcal{A}$.

The Hilbert C^*-module $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$

- Let $(\mathrm{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert $\mathcal{A}$-module. Define
$$L^2(\mathbb{R}, \mathrm{H}) = \{f : \mathbb{R} \to \mathrm{H} \mid \int_{\mathbb{R}} \|f(x)\|^2 dx < \infty\}.$$
Then we know that $L^2(\mathbb{R}, \mathrm{H})$ is a Banach space.
- For $a \in \mathcal{A}$, $f \in L^2(\mathbb{R}, \mathrm{H})$ define
$$(af)(x) = af(x).$$
Then $af \in L^2(\mathbb{R}, \mathrm{H})$. Hence $(a, f) \mapsto af$ defines a map from $\mathcal{A} \times L^2(\mathbb{R}, \mathrm{H})$ into $L^2(\mathbb{R}, \mathrm{H})$. It is easy to see that with respect to this operation $L^2(\mathbb{R}, \mathrm{H})$ becomes an $\mathcal{A}$-module.
- For $f, g \in L^2(\mathbb{R}, \mathrm{H})$ define $\langle f, g\rangle_2$ by
$$\langle f, g\rangle_2 = \int_{\mathbb{R}} \langle f(x), g(x)\rangle dx.$$
Then using Cauchy-Schwarz inequality for H and for $L^2(\mathbb{R})$ we can show that the definition is well defined.

The Hilbert C^*-module $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$

Lemma 14

Let X be a Banach space, $T \in X^*$ and $f : \mathbb{R} \to X$ be integrable. Then

$$T(\int_{\mathbb{R}} f(x)dx) = \int_{\mathbb{R}} T(f(x))dx.$$

Lemma 15

Let $\mathcal{A}$ be a unital C^*-algebra and $f : \mathbb{R} \to \mathcal{A}$ be an integrable function. Then

(i) $f(x) \geq 0$ *for a.e.* $x \in \mathbb{R}$ *implies* $\int_{\mathbb{R}} f(x)dx \geq 0$.

(ii) $x \mapsto (f(x))^*$ *is integrable and* $(\int_{\mathbb{R}} f(x)dx)^* = \int_{\mathbb{R}} f(x)^* dx$.

Theorem 16

The space $(L^2(\mathbb{R}, \mathrm{H}), \langle\cdot,\cdot\rangle_2)$ is a pre Hilbert A-module.

We shall denote $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$ to be the completion of $L^2(\mathbb{R}, \mathrm{H})$ with respect to the norm induced by the $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle_2$.

Fourier transform

Definition 17

Let $f \in L^1(\mathbb{R}, \mathrm{H})$. Then the Fourier transform of f is defined by

$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}$, where the right hand side is a Bochner integral.

Theorem 18 (Riemann-Lebesgue lemma)

If $f \in L^1(\mathbb{R}, \mathrm{H})$, then $\widehat{f} \in C_0(\mathbb{R}, \mathrm{H})$.

Theorem 19 (Fourier inversion)

Let $f \in L^1(\mathbb{R}, \mathrm{H})$ be such that $\widehat{f} \in L^1(\mathbb{R}, \mathrm{H})$, then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \widehat{f}(\xi) d\xi, \quad \text{for a.e. } x \in \mathbb{R}.$$

The main results

Theorem 20 (Plancherel theorem)

Let $f \in L^1(\mathbb{R}, \mathrm{H}) \cap L^2(\mathbb{R}, \mathrm{H})$*, then* $\langle f, f \rangle_2 = \langle \widehat{f}, \widehat{f} \rangle_2$.

Theorem 21

The Fourier transform is unitary on $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$.

- Let $f, g \in L^1(\mathbb{R}, \mathrm{H})$. Then the convolution of f and g, denoted by $f \star g$, is defined as

$$f \star g(x) = \int_{\mathbb{R}} \langle f(y), g(x-y) \rangle dy, \quad \text{a.e. } x \in \mathbb{R}.$$

Let $F(x) = \|f(x)\|$ and $G(x) = \|g(x)\|$. Then using Cauchy-Schwarz inequality we can see that $f \star g$ is well defined and $f \star g \in L^1(\mathbb{R}, \mathcal{A})$. Further as in the classical case, we can prove the following properties.

(i) If $f \in L^1(\mathbb{R}, \mathrm{H})$, $g \in L^\infty(\mathbb{R}, \mathrm{H})$ then $f \star g \in L^\infty(\mathbb{R}, \mathcal{A})$.

(ii) If $f, g \in C_c(\mathbb{R}, \mathrm{H})$ then $f \star g \in C_c(\mathbb{R}, \mathcal{A})$.

(iii) If $1 < p < \infty$, $f \in L^p(\mathbb{R}, \mathrm{H})$ and $g \in L^{p'}(\mathbb{R}, \mathrm{H})$ where $\frac{1}{p} + \frac{1}{p'} = 1$, then $f \star g \in C_0(\mathbb{R}, \mathcal{A})$.

The main results

- For $f \in L^1(\mathbb{R}, \mathcal{A})$, $g \in L^1(\mathbb{R}, \mathrm{H})$, we define

$$f \odot g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy, \quad a.e.\ x \in \mathbb{R}.$$

Then $f \odot g \in L^1(\mathbb{R}, \mathrm{H})$.

Proposition 22

If $f, g, h \in L^1(\mathbb{R}, \mathrm{H})$, then $(f \star g) \odot h \in L^1(\mathbb{R}, \mathrm{H})$.

Theorem 23

Let $f, g \in L^1(\mathbb{R}, \mathrm{H})$. *Then*

$$(f \star g)\widehat{\ }(\xi) = \langle \widehat{f}(\xi), \widehat{g}(-\xi) \rangle, \text{ for } \xi \in \mathbb{R}.$$

The main results

Theorem 24

Let $f, g, h \in L^1(\mathbb{R}, \mathrm{H})$. *Then*

$$((f \star g) \odot h)\widehat{\ }(\xi) = \langle \widehat{f}(\xi), \widehat{g}(-\xi) \rangle \widehat{h}(\xi), \text{ for } \xi \in \mathbb{R} .$$

Proposition 25

Let $T : \mathcal{L}^2(\mathbb{R}, \mathrm{H}) \to \mathcal{L}^2(\mathbb{R}, \mathrm{H})$ be bounded, linear and of the form $T = S \otimes I$, where $S \in B(L^2(\mathbb{R}))$, I is the identity operator on H such that T commutes with translations. Then there exists $m \in L^\infty(\mathbb{R})$ such that

$$(Tf)\widehat{\ }(\xi) = m(\xi)\widehat{f}(\xi), \quad f \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \quad \xi \in \mathbb{R}.$$

Remark 26

Let $m \in L^\infty(\mathbb{R})$. Define $T_m : \mathcal{L}^2(\mathbb{R}, \mathrm{H}) \to \mathcal{L}^2(\mathbb{R}, \mathrm{H})$ by

$$(T_m f)\hat{}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then

$$\|T_m f\|_2 = \|(T_m f)\hat{}\|_2 = \|m\widehat{f}\|_2 \leq \|m\|_\infty \|\widehat{f}\|_2 = \|m\|_\infty \|f\|_2.$$

Thus T_m turns out to be a continuous linear operator on $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$. Further, T_m commutes with translations. As in the classical case $L^2(\mathbb{R})$, we can say that m is a Fourier multiplier for $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$. On the other hand from Proposition 25 we assure that if $T \in B(\mathcal{L}^2(\mathbb{R}, \mathrm{H}))$ is of the form $S \otimes I$, with $S \in B(L^2(\mathbb{R}))$, I is the identity operator on H and commutes with translations, then $T = T_m$ for some $m \in L^\infty(\mathbb{R})$.

The main results

Theorem 27

Let $(\mathrm{H}, \langle \cdot, \cdot \rangle)$ *be a Hilbert* $\mathcal{A}$*-module. Let* $T : \mathcal{L}^2(\mathbb{R}, \mathrm{H}) \to \mathcal{L}^2(\mathbb{R}, \mathrm{H})$ *be a bounded, linear and surjective map satisfying*

(i) $T((f \star g) \odot h)(\xi) = \langle Tf(\xi), Tg(-\xi) \rangle Th(\xi)$, *for* $f, g, h \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \quad (f \star g) \odot h \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \; \xi \in \mathbb{R}$,

(ii) $T(\tau_s f)(\xi) = e^{-2\pi i s\xi} Tf(\xi)$, $f \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \quad s, \xi \in \mathbb{R}$,

(iii) T *is of the form* $S \otimes I$ *for some* $S \in B(L^2(\mathbb{R}))$ *and* I *is the identity operator on* H *and*

(iv) $T(-f)(-\xi) = Tf(\xi)$, $f \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \quad \xi \in \mathbb{R}$.

Then $\langle Tf(\xi), Tg(\xi) \rangle = \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle, \quad f, g \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \quad \xi \in \mathbb{R}$.

Outline of proof I

- Define $U : \mathcal{L}^2(\mathbb{R}, \mathrm{H}) \to \mathcal{L}^2(\mathbb{R}, \mathrm{H})$ by $Uf = g$ if $Tf = \widehat{g}$. Then

$$(Uf)\widehat{\ }(\xi) = \widehat{g}(\xi) = Tf(\xi).$$

- Clearly U is bounded linear and onto.
- Let $f_1, f_2, f_3 \in \mathcal{L}^2(\mathbb{R}, \mathrm{H})$ be such that $(f_1 \star f_2) \odot f_3 \in \mathcal{L}^2(\mathbb{R}, \mathrm{H})$. For $i = 1, 2, 3$ let $Uf_i = g_i$ and $U((f_1 \star f_2) \odot f_3) = g$. Then $Tf_i = \widehat{g_i}$ and $T((f_1 \star f_2) \odot f_3) = \widehat{g}, \ i = 1, 2, 3$.
- But

$$\begin{aligned} T((f_1 \star f_2) \odot f_3)(\xi) &= ((g_1 \star g_2) \odot g_3)\widehat{\ }(\xi) \\ &= ((Uf_1 \star Uf_2) \odot Uf_3)\widehat{\ }(\xi). \end{aligned}$$

- Hence $U((f_1 \star f_2) \odot f_3) = (Uf_1 \star Uf_2) \odot Uf_3$. Further, we can show that $U\tau_t = \tau_t U$ for $t \in \mathbb{R}$.

Outline of proof II

- Let $f = F \otimes u$ be a basic tensor in $\mathcal{L}^2(\mathbb{R}, \mathrm{H})$. Then

$$T(F \otimes u) = SF \otimes u.$$

- Define $S_1 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $S_1(F) = G$ if $SF = \widehat{G}$. Then

$$Uf = G \otimes u = S_1 F \otimes u = (S_1 \otimes I)(F \otimes u).$$

  Hence $U = S_1 \otimes I$.
- Using Proposition 25 there exists $m \in L^\infty(\mathbb{R})$ such that

$$(Uf)\hat{}(\xi) = m(\xi)\widehat{f}(\xi).$$

- Thus $\langle (Uf_1)\hat{}(\xi), (Uf_2)\hat{}(-\xi)\rangle (Uf_3)\hat{}(\xi) = \langle \widehat{f_1}(\xi), \widehat{f_2}(-\xi)\rangle (Uf_3)\hat{}(\xi)$.
- Using ontoness of U we can show that

$$\langle Tf_1(\xi), Tf_2(\xi)\rangle = \langle \widehat{f_1}(\xi), \widehat{f_2}(\xi)\rangle.$$

The main results

Corollary 28

If in addition to the hypotheses of Theorem 27, T satisfies $T(g_k) = g_k$ where $g_k(x) = e^{-\pi x^2} e_k$ for some countable orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of H then $Tf(\xi) = \widehat{f}(\xi), \quad f \in \mathcal{L}^2(\mathbb{R}, \mathrm{H}), \ \xi \in \mathbb{R}$.

Proof.

Let $f \in \mathcal{L}^2(\mathbb{R}, \mathrm{H})$. Then

$$\langle Tf(\xi), Tg_k(\xi)\rangle = \langle \widehat{f}(\xi), \widehat{g}_k(\xi)\rangle.$$

This is equivalent to

$$\langle Tf(\xi), e_k\rangle = \langle \widehat{f}(\xi), e_k\rangle,$$

for all $k \in \mathbb{N}$. □

The main results

Theorem 29

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space with countable orthonormal basis $\{e_k : k \in \mathbb{N}\}$ and $T : L^1(\mathbb{R}, \mathcal{H}) \to C_0(\mathbb{R}, \mathcal{H})$ be a bounded linear map satisfying

(i) $T((f \star g) \odot h)(\xi) = \langle Tf(\xi), Tg(-\xi)\rangle Th(\xi)$, *for* $f, g, h \in L^1(\mathbb{R}, \mathcal{H})$, $\xi \in \mathbb{R}$,

(ii) $T(\tau_s f)(\xi) = e^{-2\pi i s\xi} Tf(\xi)$, $f \in L^1(\mathbb{R}, \mathcal{H})$, $s, \xi \in \mathbb{R}$,

(iii) *If* $\langle f(x), e_k\rangle = 0$ *for a.e.* $x \in \mathbb{R}$ *for some* k *then* $\langle Tf(\xi), e_k\rangle = 0$ *for all* $\xi \in \mathbb{R}$ *and*

(iv) $T(-f)(-\xi) = Tf(\xi)$, $f \in L^1(\mathbb{R}, \mathcal{H})$, $\xi \in \mathbb{R}$.

Then there exists $E \subset \mathbb{R}$ *such that*

$$\langle Tf(\xi), Tg(\xi)\rangle = \chi_E(\xi)\langle \widehat{f}(\xi), \widehat{g}(\xi)\rangle, \quad f, g \in L^1(\mathbb{R}, \mathcal{H}),\ \xi \in E.$$

Outline of proof I

- Let $E = \{\xi \in \mathbb{R} : \langle Tf(\xi), e_j\rangle \neq 0,$ for some $f \in L^1(\mathbb{R}, \mathcal{H})$ and for some $j \in \mathbb{N}\}$.
- Fix $\xi \in E$. Choose $k \in \mathbb{N}$ and $h \in L^1(\mathbb{R}, \mathcal{H})$ such that $\langle Th(\xi), e_k\rangle \neq 0$.
- Define $P_k : \mathcal{H} \to \mathbb{C}$ by

$$P_k\Big(\sum_{i=1}^{\infty} \alpha_i e_i\Big) = \alpha_k.$$

- Then there exists $u_{\xi,k} \in L^\infty(\mathbb{R}, \mathcal{H})$ such that

$$P_k(Tf(\xi)) = \int_{\mathbb{R}} \langle f(t), u_{\xi,k}(t)\rangle dt.$$

- One can show that $u_{\xi,k}(t) \perp e_i$ for $a.e.\ t \in \mathbb{R},\ \ i \neq k$. Let $u_{\xi,k}(t) = \overline{v_{\xi,k}(t)} e_k$, for some $v_{\xi,k} \in L^\infty(\mathbb{R})$.

Outline of proof II

- Thus, $P_k(Tf(\xi)) = \langle \int_{\mathbb{R}} v_{\xi,k}(t)f(t)dt, e_k\rangle$.
- Now for $f, g \in L^1(\mathbb{R}, \mathcal{H})$, we obtain

$$\begin{aligned} P_k(T((f \star g) \odot h)(\xi)) &= \int_{\mathbb{R}} f \star g(s) \langle T(\tau_s h)(\xi), e_k\rangle ds \\ &= \int_{\mathbb{R}} f \star g(s) e^{-2\pi i s\xi} \langle Th(\xi), e_k\rangle ds \\ &= \langle Th(\xi), e_k\rangle \langle \widehat{f}(\xi), \widehat{g}(-\xi)\rangle. \end{aligned}$$

- But $P_k(T((f \star g) \odot h)(\xi)) = \langle Tf(\xi), Tg(-\xi)\rangle \langle Th(\xi), e_k\rangle$.
- Hence

$$\langle Tf(\xi), Tg(\xi)\rangle = \langle \widehat{f}(\xi), \widehat{g}(\xi)\rangle.$$

Corollary 30

If in addition to the hypotheses of Theorem 29, T satisfies $T(g_k) = g_k$ where $g_k(x) = e^{-\pi x^2} e_k$ for all $k \in \mathbb{N}$, then there exists $E \subset \mathbb{R}$ such that $Tf(\xi) = \chi_E(\xi)\widehat{f}(\xi), \quad f \in L^1(\mathbb{R}, \mathcal{H}), \quad \xi \in \mathbb{R}$.

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THANK YOU