Pointwise Fatou theorem and its converse for solutions of the heat equation on a stratified Lie group

17th DMHA, NISER

Jayanta Sarkar
ISI Kolkata

5th Jan, 2021

- By a measure μ we will always mean a complex or a signed Borel measure such that $|\mu|(K) < \infty$ for all compact sets K.
- If $\mu(E) \geq 0$, for all measurable sets E then μ will be called a positive measure.

Our motivation is a classical theorem of Fatou regarding boundary behavior of Poisson integrals of measures defined in the unit disc of $\mathbb{C}$. However, we prefer to present the results for the upper half space:

$$\mathbb{R}^{n+1}_{+} = \{(x, y) | x \in \mathbb{R}^n, \ y > 0\}.$$

Fatou proved his result for $n = 1$.

- By a measure μ we will always mean a complex or a signed Borel measure such that $|\mu|(K) < \infty$ for all compact sets K.
- If $\mu(E) \geq 0$, for all measurable sets E then μ will be called a positive measure.

Our motivation is a classical theorem of Fatou regarding boundary behavior of Poisson integrals of measures defined in the unit disc of $\mathbb{C}$. However, we prefer to present the results for the upper half space:

$$\mathbb{R}^{n+1}_{+} = \{(x, y) | x \in \mathbb{R}^n, \ y > 0\}.$$

Fatou proved his result for $n = 1$.

- By a measure μ we will always mean a complex or a signed Borel measure such that $|\mu|(K) < \infty$ for all compact sets K.
- If $\mu(E) \geq 0$, for all measurable sets E then μ will be called a positive measure.

Our motivation is a classical theorem of Fatou regarding boundary behavior of Poisson integrals of measures defined in the unit disc of $\mathbb{C}$. However, we prefer to present the results for the upper half space:

$$\mathbb{R}^{n+1}_{+} = \{(x, y) | x \in \mathbb{R}^n, \ y > 0\}.$$

Fatou proved his result for $n = 1$.

- By a measure μ we will always mean a complex or a signed Borel measure such that $|\mu|(K) < \infty$ for all compact sets K.
- If $\mu(E) \geq 0$, for all measurable sets E then μ will be called a positive measure.

Our motivation is a classical theorem of Fatou regarding boundary behavior of Poisson integrals of measures defined in the unit disc of $\mathbb{C}$. However, we prefer to present the results for the upper half space:

$$\mathbb{R}^{n+1}_+ = \{(x, y)|x \in \mathbb{R}^n,\ y > 0\}.$$

Fatou proved his result for $n = 1$.

Poisson kernel

The Poisson kernel of $\mathbb{R}^{n+1}_+$:

$$P(x,y) = y^{-n}P(x/y,1) = c_n \frac{y}{(y^2+\|x\|^2)^{\frac{n+1}{2}}}, \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

The Poisson integral $P[\mu]$ of μ:

$$P[\mu](x,y) = \int_{\mathbb{R}^n} P(x-\xi,y)\, d\mu(\xi),$$

whenever the integral above converges absolutely for $(x,y) \in \mathbb{R}^{n+1}_+$.

Poisson kernel

The Poisson kernel of $\mathbb{R}^{n+1}_+$:

$$P(x,y) = y^{-n}P(x/y,1) = c_n \frac{y}{(y^2 + \|x\|^2)^{\frac{n+1}{2}}}, \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

The Poisson integral $P[\mu]$ of μ:

$$P[\mu](x,y) = \int_{\mathbb{R}^n} P(x-\xi, y)\, d\mu(\xi),$$

whenever the integral above converges absolutely for $(x,y) \in \mathbb{R}^{n+1}_+$.

Poisson kernel

The Poisson kernel of $\mathbb{R}^{n+1}_+$:

$$P(x,y) = y^{-n}P(x/y,1) = c_n \frac{y}{(y^2+\|x\|^2)^{\frac{n+1}{2}}}, \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

The Poisson integral $P[\mu]$ of μ:

$$P[\mu](x,y) = \int_{\mathbb{R}^n} P(x-\xi, y)\, d\mu(\xi),$$

whenever the integral above converges absolutely for $(x,y) \in \mathbb{R}^{n+1}_+$.

Fatou's theorem

Theorem (Fatou, 1906)

Suppose that μ is a measure on $\mathbb{R}$ with distribution function F. If $F'(x_0) = L$, then $P[\mu]$ has nontangential limit L at x_0, that is,

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} P[\mu](x,y) = L,$$

for all $\alpha > 0$.

Result of Loomis

- As shown by Loomis [Loo43], converse of the theorem above fails in general but it holds true if μ is positive.
- If u is a positive harmonic function in $\mathbb{R}^{n+1}_+$, then $\exists$ unique $C \geq 0$, $\mu \geq 0$ such that

$$u(x,y) = Cy + P[\mu](x,y), \quad (x,y) \in \mathbb{R}^{n+1}_+,$$

Theorem (Loomis)

If u is a positive harmonic function in $\mathbb{R}^2_+$, then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \implies F'(x_0) = L.$$

Result of Loomis

- As shown by Loomis [Loo43], converse of the theorem above fails in general but it holds true if μ is positive.
- If u is a positive harmonic function in $\mathbb{R}^{n+1}_+$, then $\exists$ unique $C \geq 0$, $\mu \geq 0$ such that

$$u(x,y) = Cy + P[\mu](x,y), \quad (x,y) \in \mathbb{R}^{n+1}_+,$$

Theorem (Loomis)

If u is a positive harmonic function in $\mathbb{R}^2_+$, then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \implies F'(x_0) = L.$$

Result of Loomis

- As shown by Loomis [Loo43], converse of the theorem above fails in general but it holds true if μ is positive.
- If u is a positive harmonic function in $\mathbb{R}^{n+1}_+$, then $\exists$ unique $C \geq 0$, $\mu \geq 0$ such that

$$u(x,y) = Cy + P[\mu](x,y), \quad (x,y) \in \mathbb{R}^{n+1}_+,$$

Theorem (Loomis)

If u is a positive harmonic function in $\mathbb{R}^2_+$, then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \implies F'(x_0) = L.$$

Result of Ramey-Ullrich

- Higher dimensional interpretation of F' is a issue here. Ramey and Ullrich interpreted F' as the strong derivative of μ.
- The measure μ on $\mathbb{R}^n$ has strong derivative $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if

$$\lim_{r\to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L = D\mu(x_0),$$

holds for every open ball B, where $rB = \{rx \mid x \in B\}$, $r > 0$.

Theorem (Ramey-Ullrich[RU88])

Suppose that u is positive and harmonic in $\mathbb{R}^{n+1}_+$ with boundary measure μ. Then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \iff D\mu(x_0) = L.$$

Result of Ramey-Ullrich

- Higher dimensional interpretation of F' is a issue here. Ramey and Ullrich interpreted F' as the strong derivative of μ.
- The measure μ on $\mathbb{R}^n$ has strong derivative $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if

$$\lim_{r \to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L = D\mu(x_0),$$

holds for every open ball B, where $rB = \{rx \mid x \in B\}$, $r > 0$.

Theorem (Ramey-Ullrich[RU88])

Suppose that u is positive and harmonic in $\mathbb{R}^{n+1}_+$ with boundary measure μ. Then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \iff D\mu(x_0) = L.$$

Result of Ramey-Ullrich

- Higher dimensional interpretation of F' is a issue here. Ramey and Ullrich interpreted F' as the strong derivative of μ.
- The measure μ on $\mathbb{R}^n$ has strong derivative $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if

$$\lim_{r\to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L = D\mu(x_0),$$

holds for every open ball B, where $rB = \{rx \mid x \in B\}$, $r > 0$.

Theorem (Ramey-Ullrich[RU88])

Suppose that u is positive and harmonic in $\mathbb{R}^{n+1}_+$ with boundary measure μ. Then

$$\lim_{\substack{(x,y)\to(x_0,0)\\ \|x-x_0\|<\alpha y}} u(x,y) = L, \text{ for all } \alpha > 0 \iff D\mu(x_0) = L.$$

Heat equation in $\mathbb{R}^{n+1}_+$

- The heat equation in $\mathbb{R}^{n+1}_+$:
  $\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+$.
- The heat kernel (aka the Gauss-Weierstrass kernel):
  $W(x,t) = w_{\sqrt{t}}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+$, where
  $w(x) = W(x,1)$.
- The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$:
  $W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y)$,
  whenever the integral exists.
- If $W[|\mu|](x_0,t_0) < \infty$, then $W[\mu](x,t)$ exists in the strip
  $\mathbb{R}^n \times (0,t_0)$ and solves the heat equation there.

Heat equation in $\mathbb{R}^{n+1}_+$

- The heat equation in $\mathbb{R}^{n+1}_+$:
  $\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+$.
- The heat kernel (aka the Gauss-Weierstrass kernel):
  $W(x,t) = w_{\sqrt{t}}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+$, where
  $w(x) = W(x,1)$.
- The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$:
  $W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y)$,
  whenever the integral exists.
- If $W[|\mu|](x_0,t_0) < \infty$, then $W[\mu](x,t)$ exists in the strip $\mathbb{R}^n \times (0,t_0)$ and solves the heat equation there.

Heat equation in $\mathbb{R}^{n+1}_+$

- The heat equation in $\mathbb{R}^{n+1}_+$:
  $\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+$.
- The heat kernel (aka the Gauss-Weierstrass kernel):
  $W(x,t) = w_{\sqrt{t}}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+$, where $w(x) = W(x,1)$.
- The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$:
  $W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y)$,
  whenever the integral exists.
- If $W[|\mu|](x_0,t_0) < \infty$, then $W[\mu](x,t)$ exists in the strip $\mathbb{R}^n \times (0,t_0)$ and solves the heat equation there.

Heat equation in $\mathbb{R}^{n+1}_+$

- The heat equation in $\mathbb{R}^{n+1}_+$:
  $\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+$.
- The heat kernel (aka the Gauss-Weierstrass kernel):
  $W(x,t) = w_{\sqrt{t}}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+$, where $w(x) = W(x,1)$.
- The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$:
  $W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y)$,
  whenever the integral exists.
- If $W[|\mu|](x_0,t_0) < \infty$, then $W[\mu](x,t)$ exists in the strip $\mathbb{R}^n \times (0,t_0)$ and solves the heat equation there.

Result of Gehring

- As $W(x,t) = w_{\sqrt{t}}(x)$, it is natural to investigate the boundary behavior of solutions of the heat equation in the parabolic region: $\mathrm{P}(x_0, \alpha) = \{(x,t) \in \mathbb{R}^{n+1}_{+} \mid \|x - x_0\|^2 < \alpha t\}$.

Theorem (Gehring[Geh60], 1960)

Suppose μ is a measure on $\mathbb{R}$ with distribution function F such that $W[\mu]$ is well-defined in $\mathbb{R} \times (0, t_0)$.

i) *If $F'(x_0) = L$, then $W[\mu]$ has parabolic limit L at x_0, that is $\lim_{\substack{(x,y)\to(x_0,0)\\ (x,t)\in P(x_0,\alpha)}} W[\mu](x,t) = L$, for all $\alpha > 0$.*

ii) *Further assume that μ is positive. If $W[\mu]$ has parabolic limit L at x_0, then $F'(x_0) = L$.*

Result of Gehring

- As $W(x,t) = w_{\sqrt{t}}(x)$, it is natural to investigate the boundary behavior of solutions of the heat equation in the parabolic region: $\mathrm{P}(x_0,\alpha) = \{(x,t) \in \mathbb{R}^{n+1}_+ \mid \|x - x_0\|^2 < \alpha t\}$.

Theorem (Gehring[Geh60], 1960)

Suppose μ is a measure on $\mathbb{R}$ with distribution function F such that $W[\mu]$ is well-defined in $\mathbb{R} \times (0, t_0)$.

i) *If $F'(x_0) = L$, then $W[\mu]$ has parabolic limit L at x_0, that is* $\lim_{\substack{(x,y)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} W[\mu](x,t) = L$, *for all* $\alpha > 0$.

ii) *Further assume that μ is positive. If $W[\mu]$ has parabolic limit L at x_0, then $F'(x_0) = L$.*

Result of Gehring

- As $W(x,t) = w_{\sqrt{t}}(x)$, it is natural to investigate the boundary behavior of solutions of the heat equation in the parabolic region: $\mathrm{P}(x_0,\alpha) = \{(x,t) \in \mathbb{R}^{n+1}_+ \mid \|x - x_0\|^2 < \alpha t\}$.

Theorem (Gehring[Geh60], 1960)

Suppose μ is a measure on $\mathbb{R}$ with distribution function F such that $W[\mu]$ is well-defined in $\mathbb{R} \times (0, t_0)$.

i) *If $F'(x_0) = L$, then $W[\mu]$ has parabolic limit L at x_0, that is $\lim_{\substack{(x,y)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} W[\mu](x,t) = L$, for all $\alpha > 0$.*

ii) *Further assume that μ is positive. If $W[\mu]$ has parabolic limit L at x_0, then $F'(x_0) = L$.*

Parabolic convergence of solution of the heat equation in $\mathbb{R}^{n+1}_+$

- If u is a positive solution then $u = W[\mu]$, for some $\mu \geq 0$ on $\mathbb{R}^n$.

Theorem

Suppose μ is as above. Then u has parabolic limit L at x_0, that is $\lim_{\substack{(x,t)\to(x_0,0)\\ (x,t)\in P(x_0,\alpha)}} u(x,t) = L$, for all $\alpha > 0$, if and only if $D\mu(x_0) = L$.

- In fact, Brossard and Chevalier[BC90] proved the above theorem for measures μ satisfying $\sup_{(x,t)\in B(0,1)\times(0,t_0)} \left(W[|\mu|](x,t) - |W[\mu](x,t)|\right) < \infty$.

Parabolic convergence of solution of the heat equation in $\mathbb{R}^{n+1}_+$

- If u is a positive solution then $u = W[\mu]$, for some $\mu \geq 0$ on $\mathbb{R}^n$.

Theorem

Suppose μ is as above. Then u has parabolic limit L at x_0, that is $\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} u(x,t) = L$, for all $\alpha > 0$, if and only if $D\mu(x_0) = L$.

- In fact, Brossard and Chevalier[BC90] proved the above theorem for measures μ satisfying $\sup_{(x,t)\in B(0,1)\times(0,t_0)} \left(W[|\mu|](x,t) - |W[\mu](x,t)|\right) < \infty$.

Parabolic convergence of solution of the heat equation in $\mathbb{R}^{n+1}_+$

- If u is a positive solution then $u = W[\mu]$, for some $\mu \geq 0$ on $\mathbb{R}^n$.

Theorem

Suppose μ is as above. Then u has parabolic limit L at x_0, that is $\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} u(x,t) = L$, *for all* $\alpha > 0$, *if and only if* $D\mu(x_0) = L$.

• In fact, Brossard and Chevalier[BC90] proved the above theorem for measures μ satisfying
$\sup_{(x,t)\in B(0,1)\times(0,t_0)} \left(W[|\mu|](x,t) - |W[\mu](x,t)| \right) < \infty$.

Stratified Lie groups

A stratified Lie group $(G, \circ)$ is a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_l$, such that

$$[V_1, V_j] = V_{j+1}, \; 1 \leq j < l, \qquad [V_1, V_l] = 0.$$

$[V_1, V_j] = \mathrm{span}_{\mathbb{R}} \{[X, Y] \mid X \in V_1, Y \in V_j\}$.

The Lie algebra $\mathfrak{g}$ is eqquiped with a cannonical family of dilations $\{\delta_r \mid r \in (0, \infty)\}$, which are Lie algebra automophisms defined by

$$\delta_r \left(\sum_{j=1}^{l} X_j \right) = \sum_{j=1}^{l} r^j X_j, \quad X_j \in V_j.$$

Stratified Lie groups

A stratified Lie group $(G, \circ)$ is a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_l$, such that

$$[V_1, V_j] = V_{j+1}, \ 1 \leq j < l, \qquad [V_1, V_l] = 0.$$

$[V_1, V_j] = \text{span}_{\mathbb{R}} \{[X, Y] \mid X \in V_1, Y \in V_j\}$.
The Lie algebra $\mathfrak{g}$ is eqquiped with a cannonical family of dilations $\{\delta_r \mid r \in (0, \infty)\}$, which are Lie algebra automophisms defined by

$$\delta_r \left(\sum_{j=1}^{l} X_j \right) = \sum_{j=1}^{l} r^j X_j, \quad X_j \in V_j.$$

Homogeneous norm

A stratified Lie group G always admits a homogeneous norm which is a continuous function $d : G \to [0, \infty)$, satisfying the following:

i) d is smooth on $G \setminus \{0\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0, \infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = 0$.

- $d(x \circ y) \leq C(d(x) + d(y))$.
- $C^{-1} d_1(x) \leq d_2(x) \leq C d_1(x)$.

d-ball: $B_d(x, s) = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}$.

Fact: $m\left(B_d(x, s)\right) = s^Q m(B(\underline{0}, 1))$, where $Q = \sum_{j=1}^{l} j \, dim \, V_j$.

Homogeneous norm

A stratified Lie group G always admits a homogeneous norm which is a continuous function $d : G \to [0, \infty)$, satisfying the following:

i) d is smooth on $G \setminus \{0\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0, \infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = 0$.

- $d(x \circ y) \leq C(d(x) + d(y))$.
- $C^{-1} d_1(x) \leq d_2(x) \leq C d_1(x)$.

d-ball: $B_d(x, s) = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}$.

Fact: $m(B_d(x, s)) = s^Q m(B(\underline{0}, 1))$, where $Q = \sum_{j=1}^{l} j \, dim \, V_j$.

Homogeneous norm

A stratified Lie group G always admits a homogeneous norm which is a continuous function $d : G \to [0, \infty)$, satisfying the following:

i) d is smooth on $G \setminus \{0\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0, \infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = 0$.

- $d(x \circ y) \leq C(d(x) + d(y))$.
- $C^{-1} d_1(x) \leq d_2(x) \leq C d_1(x)$.

d-ball: $B_d(x, s) = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}$.

Fact: $m\left(B_d(x, s)\right) = s^Q m(B(\underline{0}, 1))$, where $Q = \sum_{j=1}^{l} j \, dim \, V_j$.

Homogeneous norm

A stratified Lie group G always admits a homogeneous norm which is a continuous function $d : G \to [0, \infty)$, satisfying the following:

i) d is smooth on $G \setminus \{0\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0, \infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = 0$.

- $d(x \circ y) \leq C(d(x) + d(y))$.
- $C^{-1}d_1(x) \leq d_2(x) \leq Cd_1(x)$.

d-ball: $B_d(x, s) = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}$.

Fact: $m\left(B_d(x, s)\right) = s^Q m(B(\underline{0}, 1))$, where $Q = \sum_{j=1}^{l} j \, dim \, V_j$.

Homogeneous norm

A stratified Lie group G always admits a homogeneous norm which is a continuous function $d : G \to [0, \infty)$, satisfying the following:

i) d is smooth on $G \setminus \{0\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0, \infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = 0$.

- $d(x \circ y) \leq C(d(x) + d(y))$.
- $C^{-1} d_1(x) \leq d_2(x) \leq C d_1(x)$.

d-ball: $B_d(x, s) = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}$.

Fact: $m(B_d(x, s)) = s^Q m(B(\underline{0}, 1))$, where $Q = \sum_{j=1}^{l} j \, dim \, V_j$.

Sub-Laplacian

We identify $\mathfrak{g}$ as the Lie algebra of all left G-invariant vector fields on G and fix a basis $\{X_1, X_2, \cdots, X_{N_1}\}$ for V_1, which generates $\mathfrak{g}$ as a Lie algebra. The second order differential operator $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ is called a sub-Laplacian on G.

- There exists a homogeneous norm $d_{\mathcal{L}}$ on G such that $d_{\mathcal{L}}(\cdot)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.
- $\mathcal{H} = \mathcal{L} - \frac{\partial}{\partial t}$ is the heat operator associated to the sub-Laplacian $\mathcal{L}$.
- Since $\{X_1, X_2, \cdots, X_{N_1}\}$ generates $\mathfrak{g}$ as a Lie algebra, Hörmander's theorem $\implies$ $\mathcal{L}$ and $\mathcal{H}$ are hypoelliptic.

Sub-Laplacian

We identify $\mathfrak{g}$ as the Lie algebra of all left G-invariant vector fields on G and fix a basis $\{X_1, X_2, \cdots, X_{N_1}\}$ for V_1, which generates $\mathfrak{g}$ as a Lie algebra. The second order differential operator $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ is called a sub-Laplacian on G.

- There exists a homogeneous norm $d_{\mathcal{L}}$ on G such that $d_{\mathcal{L}}(\cdot)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.
- $\mathcal{H} = \mathcal{L} - \frac{\partial}{\partial t}$ is the heat operator associated to the sub-Laplacian $\mathcal{L}$.
- Since $\{X_1, X_2, \cdots, X_{N_1}\}$ generates $\mathfrak{g}$ as a Lie algebra, Hörmander's theorem $\implies$ $\mathcal{L}$ and $\mathcal{H}$ are hypoelliptic.

Sub-Laplacian

We identify $\mathfrak{g}$ as the Lie algebra of all left G-invariant vector fields on G and fix a basis $\{X_1, X_2, \cdots, X_{N_1}\}$ for V_1, which generates $\mathfrak{g}$ as a Lie algebra. The second order differential operator $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ is called a sub-Laplacian on G.

- There exists a homogeneous norm $d_{\mathcal{L}}$ on G such that $d_{\mathcal{L}}(\cdot)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.
- $\mathcal{H} = \mathcal{L} - \frac{\partial}{\partial t}$ is the heat operator associated to the sub-Laplacian $\mathcal{L}$.
- Since $\{X_1, X_2, \cdots, X_{N_1}\}$ generates $\mathfrak{g}$ as a Lie algebra, Hörmander's theorem $\implies$ $\mathcal{L}$ and $\mathcal{H}$ are hypoelliptic.

Sub-Laplacian

We identify $\mathfrak{g}$ as the Lie algebra of all left G-invariant vector fields on G and fix a basis $\{X_1, X_2, \cdots, X_{N_1}\}$ for V_1, which generates $\mathfrak{g}$ as a Lie algebra. The second order differential operator $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ is called a sub-Laplacian on G.

- There exists a homogeneous norm $d_{\mathcal{L}}$ on G such that $d_{\mathcal{L}}(\cdot)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.
- $\mathcal{H} = \mathcal{L} - \frac{\partial}{\partial t}$ is the heat operator associated to the sub-Laplacian $\mathcal{L}$.
- Since $\{X_1, X_2, \cdots, X_{N_1}\}$ generates $\mathfrak{g}$ as a Lie algebra, Hörmander's theorem $\implies$ $\mathcal{L}$ and $\mathcal{H}$ are hypoelliptic.

Heisenberg group

The simplest nontrivial example of a stratified Lie group is the Heisenberg group $H^n = \mathbb{R}^{2n} \oplus \mathbb{R}$.

A basis for $\mathbb{R}^{2n}$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \; Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

$\{X_j, Y_j\}_{1 \leq j \leq n}$ generates the Lie algebra of H^n as

$$[X_j, Y_j] = -4\frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

The sub-Laplacian $\sum_{j=1}^{n} X_j^2 + Y_j^2$ also known as the Kohn Laplacian.

Heisenberg group

The simplest nontrivial example of a stratified Lie group is the Heisenberg group $H^n = \mathbb{R}^{2n} \oplus \mathbb{R}$.
A basis for $\mathbb{R}^{2n}$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

$\{X_j, Y_j\}_{1 \leq j \leq n}$ generates the Lie algebra of H^n as

$$[X_j, Y_j] = -4\frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

The sub-Laplacian $\sum_{j=1}^{n} X_j^2 + Y_j^2$ also known as the Kohn Laplacian.

Heisenberg group

The simplest nontrivial example of a stratified Lie group is the Heisenberg group $H^n = \mathbb{R}^{2n} \oplus \mathbb{R}$.
A basis for $\mathbb{R}^{2n}$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \;\; Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

$\{X_j, Y_j\}_{1 \leq j \leq n}$ generates the Lie algebra of H^n as

$$[X_j, Y_j] = -4\frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

The sub-Laplacian $\sum_{j=1}^{n} X_j^2 + Y_j^2$ also known as the Kohn Laplacian.

Heisenberg group

The simplest nontrivial example of a stratified Lie group is the Heisenberg group $H^n = \mathbb{R}^{2n} \oplus \mathbb{R}$.
A basis for $\mathbb{R}^{2n}$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

$\{X_j, Y_j\}_{1 \leq j \leq n}$ generates the Lie algebra of H^n as

$$[X_j, Y_j] = -4 \frac{\partial}{\partial s}, \quad 1 \leq j \leq n.$$

The sub-Laplacian $\sum_{j=1}^{n} X_j^2 + Y_j^2$ also known as the Kohn Laplacian.

Heat kernel on stratified Lie groups

The heat equation on G: $\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0,\infty)$.

The fundamental solution (heat kernel) of $\mathcal{H}$:

$\Gamma(x,t;\xi) := \Gamma(\xi^{-1} \circ x, t)$, where Γ is a smooth, strictly positive function on $G \times (0,\infty)$ satisfying the following properties:

(i) $\Gamma(x,t) = \Gamma(x^{-1}, t)$;

(ii) $\Gamma(\delta_r(x), r^2 t) = r^{-Q}\Gamma(x,t)$;

(iii) $\int_G \Gamma(x,t)\, dm(x) = 1$;

(iv) [Bonfiglioli et al., 2002] Following Gaussian estimates hold.

$$c_0^{-1} t^{-\frac{Q}{2}} \exp\left(-\frac{c_0 d_{\mathcal{L}}(x)^2}{t}\right) \leq \Gamma(x,t) \leq c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right).$$

Heat kernel on stratified Lie groups

The heat equation on G: $\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0,\infty)$.

The fundamental solution (heat kernel) of $\mathcal{H}$:

$\Gamma(x,t;\xi) := \Gamma(\xi^{-1} \circ x, t)$, where Γ is a smooth, strictly positive function on $G \times (0,\infty)$ satisfying the following properties:

(i) $\Gamma(x,t) = \Gamma(x^{-1}, t)$;

(ii) $\Gamma(\delta_r(x), r^2 t) = r^{-Q}\Gamma(x,t)$;

(iii) $\int_G \Gamma(x,t)\, dm(x) = 1$;

(iv) [Bonfiglioli et al., 2002] Following Gaussian estimates hold.

$$c_0^{-1} t^{-\frac{Q}{2}} \exp\left(-\frac{c_0 d_{\mathcal{L}}(x)^2}{t}\right) \leq \Gamma(x,t) \leq c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right).$$

Heat kernel on stratified Lie groups

The heat equation on G: $\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0,\infty)$.

The fundamental solution (heat kernel) of $\mathcal{H}$:

$\Gamma(x,t;\xi) := \Gamma(\xi^{-1} \circ x, t)$, where Γ is a smooth, strictly positive function on $G \times (0,\infty)$ satisfying the following properties:

(i) $\Gamma(x,t) = \Gamma(x^{-1},t)$;

(ii) $\Gamma(\delta_r(x), r^2 t) = r^{-Q}\Gamma(x,t)$;

(iii) $\int_G \Gamma(x,t)\, dm(x) = 1$;

(iv) [Bonfiglioli et al., 2002] Following Gaussian estimates hold.

$$c_0^{-1} t^{-\frac{Q}{2}} \exp\left(-\frac{c_0 d_{\mathcal{L}}(x)^2}{t}\right) \leq \Gamma(x,t) \leq c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right).$$

Heat kernel on stratified Lie groups

The heat equation on G: $\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0,\infty)$.

The fundamental solution (heat kernel) of $\mathcal{H}$:

$\Gamma(x,t;\xi) := \Gamma(\xi^{-1} \circ x, t)$, where Γ is a smooth, strictly positive function on $G \times (0,\infty)$ satisfying the following properties:

(i) $\Gamma(x,t) = \Gamma(x^{-1},t)$;

(ii) $\Gamma(\delta_r(x), r^2 t) = r^{-Q}\Gamma(x,t)$;

(iii) $\int_G \Gamma(x,t)\, dm(x) = 1$;

(iv) [Bonfiglioli et al., 2002] Following Gaussian estimates hold.

$$c_0^{-1} t^{-\frac{Q}{2}} \exp\left(-\frac{c_0 d_{\mathcal{L}}(x)^2}{t}\right) \leq \Gamma(x,t) \leq c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right).$$

Parabolic convergence and strong derivative

- A function u defined on $G \times (0, t_0)$, for some $t_0 \in (0, \infty]$, is said to have parabolic limit $L \in \mathbb{C}$, at $x_0 \in G$, if for each $\alpha \in (0, \infty)$

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(x_0, \alpha)}} u(x,t) = L,$$

where $\mathrm{P}(x_0, \alpha) = \{(x,t) \in G \times (0, \infty) \mid (d_{\mathcal{L}}(x_0^{-1} \circ x))^2 < \alpha t\}$.

- Given a measure μ on G, we say that μ has strong derivative $L \in \mathbb{C}$, at $x_0 \in G$, if

$$\lim_{r \to 0} \frac{\mu(x_0 \circ \delta_r(B))}{m(x_0 \circ \delta_r(B))} = L,$$

holds for every $d_{\mathcal{L}}$-ball B in G.

Parabolic convergence and strong derivative

- A function u defined on $G \times (0, t_0)$, for some $t_0 \in (0, \infty]$, is said to have parabolic limit $L \in \mathbb{C}$, at $x_0 \in G$, if for each $\alpha \in (0, \infty)$

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(x_0, \alpha)}} u(x,t) = L,$$

where $\mathrm{P}(x_0, \alpha) = \{(x,t) \in G \times (0, \infty) \mid (d_{\mathcal{L}}(x_0^{-1} \circ x))^2 < \alpha t\}$.

- Given a measure μ on G, we say that μ has strong derivative $L \in \mathbb{C}$, at $x_0 \in G$, if

$$\lim_{r \to 0} \frac{\mu(x_0 \circ \delta_r(B))}{m(x_0 \circ \delta_r(B))} = L,$$

holds for every $d_{\mathcal{L}}$-ball B in G.

Main result

Lemma (Bonfiglioli-Uguzzoni[BU05], 2005)

Let $u \geq 0$ be a solution of the heat equation in the strip $G \times (0, T)$. Then there exists a unique positive measure μ on G such that
$u(x,t) = \Gamma[\mu](x,t) = \int_G \Gamma(\xi^{-1} \circ x, t)\, d\mu(\xi), \quad (x,t) \in G \times (0,T).$

Theorem (_,[Sar21a])

Suppose that u is as above, and that $x_0 \in G$, $L \in [0, \infty)$. Then the following statements are equivalent.

(i) *u has parabolic limit L at x_0.*

(ii) *μ has strong derivative L at x_0.*

Main result

Lemma (Bonfiglioli-Uguzzoni[BU05], 2005)

Let $u \geq 0$ be a solution of the heat equation in the strip $G \times (0, T)$. Then there exists a unique positive measure μ on G such that
$u(x,t) = \Gamma[\mu](x,t) = \int_G \Gamma(\xi^{-1} \circ x, t)\, d\mu(\xi), \quad (x,t) \in G \times (0,T).$

Theorem (_,[Sar21a])

Suppose that u is as above, and that $x_0 \in G$, $L \in [0, \infty)$. Then the following statements are equivalent.

(i) *u has parabolic limit L at x_0.*

(ii) *μ has strong derivative L at x_0.*

Some auxilary results I

- Suppose $\{\mu_j \mid j \in \mathbb{N}\}$, μ are positive measures. If $\Gamma[\mu_j] \to \Gamma[\mu]$ normally, i.e, uniformly on every compact set in $G \times (0, \infty)$, then

$$\int_G f \, d\mu_j \to \int_G f \, d\mu$$

for all $f \in C_c(G)$.

- Suppose that $L \in [0, \infty)$, and that

$$\int_G f \, d\mu_j \to \int_G f \, d\mu,$$

for all $f \in C_c(G)$. Then $\mu = Lm$ if and only if $\mu_j(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.

Some auxilary results I

- Suppose $\{\mu_j \mid j \in \mathbb{N}\}$, μ are positive measures. If $\Gamma[\mu_j] \to \Gamma[\mu]$ normally, i.e, uniformly on every compact set in $G \times (0, \infty)$, then

$$\int_G f \, d\mu_j \to \int_G f \, d\mu$$

for all $f \in C_c(G)$.

- Suppose that $L \in [0, \infty)$, and that

$$\int_G f \, d\mu_j \to \int_G f \, d\mu,$$

for all $f \in C_c(G)$. Then $\mu = Lm$ if and only if $\mu_j(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.

Some auxilary results II

- $M_{HL}(\mu)(x_0) = \sup_{r>0} \mu(B(x_0, r))/m(B(x_0, r))$.
  Maximal inequality:
  $c_n M_{HL}(\mu)(x_0) \leq \sup_{t>0} \Gamma[\mu](x_0, t^2) \leq \sup_{(x,t)\in \mathrm{P}(x_0,\alpha)} \Gamma[\mu](x, t) \leq c_\alpha M_{HL}(\mu)(x_0)$,
- Let $\{u_j\}$ be a sequence of solutions of the heat equation in $G \times (0, \infty)$. If $\{u_j\}$ is locally bounded, then it has a subsequence which converges to some solution v normally. This result was recently proved by Bär[BÏ3].

Some auxilary results II

- $M_{HL}(\mu)(x_0) = \sup_{r>0} \mu(B(x_0,r))/m(B(x_0,r))$.
  Maximal inequality:
  $c_n M_{HL}(\mu)(x_0) \leq \sup_{t>0} \Gamma[\mu](x_0,t^2) \leq \sup_{(x,t)\in \mathrm{P}(x_0,\alpha)} \Gamma[\mu](x,t) \leq c_\alpha M_{HL}(\mu)(x_0)$,
- Let $\{u_j\}$ be a sequence of solutions of the heat equation in $G \times (0,\infty)$. If $\{u_j\}$ is locally bounded, then it has a subsequence which converges to some solution v normally. This result was recently proved by Bär[B13].

Some auxilary results II

- $M_{HL}(\mu)(x_0) = \sup_{r>0} \mu(B(x_0, r))/m(B(x_0, r))$.
  Maximal inequality:
  $c_n M_{HL}(\mu)(x_0) \leq \sup_{t>0} \Gamma[\mu](x_0, t^2) \leq \sup_{(x,t)\in \mathrm{P}(x_0,\alpha)} \Gamma[\mu](x, t) \leq c_\alpha M_{HL}(\mu)(x_0)$,
- Let $\{u_j\}$ be a sequence of solutions of the heat equation in $G \times (0, \infty)$. If $\{u_j\}$ is locally bounded, then it has a subsequence which converges to some solution v normally. This result was recently proved by Bär[B13].

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right) / m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty,\ u(0, t^2) \to L$ as $t \to 0$
  $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0, t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$.Using corresponding r_j's, define $u_j(x, t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right) / m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty$, $u(0, t^2) \to L$ as $t \to 0$ $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0, t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$.Using corresponding r_j's, define $u_j(x, t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right) / m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty,\ u(0, t^2) \to L$ as $t \to 0$ $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0, t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$.Using corresponding r_j's, define $u_j(x, t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right)/m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty,\ u(0,t^2) \to L$ as $t \to 0$
  $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0,t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$.Using corresponding r_j's, define $u_j(x,t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right)/m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty,\ u(0, t^2) \to L$ as $t \to 0$
  $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0, t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$. Using corresponding r_j's, define $u_j(x, t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Sketch of the proof

Step 1:

- One can reduce to the case $x_0 = 0$, and $\mu(\mathbb{R}^n) < \infty$.
- Fix a $d_{\mathcal{L}}$-ball B_0, a sequence $\{r_j\}$ of positive numbers with $r_j \to 0$. Set $L_j = \mu\left(\delta_{r_j}(B_0)\right) / m\left(\delta_{r_j}(B_0)\right)$.
- $\mu(\mathbb{R}^n) < \infty,\ u(0, t^2) \to L$ as $t \to 0$
  $\implies L_j \leq CM_{HL}(\mu)(0) \leq C' \sup_{0<t<\infty} \Gamma[\mu](0, t^2) < \infty$.
- Take convergent subsequence of $\{L_j\}$ and denote it also by $\{L_j\}$.Using corresponding r_j's, define $u_j(x, t) = u\left(\delta_{r_j}(x), r_j^2 t\right)$.

Step 2:

- $\mathrm{P}(0, \alpha)$ is invariant under the action $(r, (x, t)) \to (\delta_{r_j}(x), r_j^2 t)$.
- $\{u_j\}$ is a sequence of solutions with $\sup_j \|u_j\|_{L^\infty(K)} < \infty$, for all compact set $K \subset G \times (0, \infty)$.
- Use Bär's result to get a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \to v$ normally.
- Fix (x_0, t_0). Choose $\eta > 0$ such that $(x_0, t_0) \in \mathrm{P}(0, \alpha)$. Since $r_{j_k} \to 0$, $v(x_0, t_0) = \lim_{k\to\infty} u\left(\delta_{r_{j_k}}(x_0), r_{j_k}^2 t_0\right) = L$.

Step 2:

- $\mathrm{P}(0, \alpha)$ is invariant under the action $(r, (x, t)) \to (\delta_{r_j}(x), r_j^2 t)$.
- $\{u_j\}$ is a sequence of solutions with $\sup_j \|u_j\|_{L^\infty(K)} < \infty$, for all compact set $K \subset G \times (0, \infty)$.
- Use Bär's result to get a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \to v$ normally.
- Fix (x_0, t_0). Choose $\eta > 0$ such that $(x_0, t_0) \in \mathrm{P}(0, \alpha)$. Since $r_{j_k} \to 0$, $v(x_0, t_0) = \lim_{k\to\infty} u\left(\delta_{r_{j_k}}(x_0), r_{j_k}^2 t_0\right) = L$.

Step 3:

- Set $\mu_k(E) = r_{j_k}^{-Q}\mu(\delta_{r_{j_k}}(E))$. Then $u_{j_k} = \Gamma[\mu_k] \to L = \Gamma[Lm]$ normally.
- $\mu_{r_{j_k}}(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.
- Put $B = B_0$ to conclude that $L_{j_k} = \mu(\delta_{r_{j_k}}(B_0))/m(\delta_{r_{j_k}}(B_0)) \to L$.
- Every convergent subsequence of the bounded sequence $\{L_j\}$ has limit L. Hence, $L_j \to L$.
- μ has strong derivative L at 0.

Step 3:

- Set $\mu_k(E) = r_{j_k}^{-Q}\mu(\delta_{r_{j_k}}(E))$. Then $u_{j_k} = \Gamma[\mu_k] \to L = \Gamma[Lm]$ normally.
- $\mu_{r_{j_k}}(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.
- Put $B = B_0$ to conclude that $L_{j_k} = \mu(\delta_{r_{j_k}}(B_0))/m(\delta_{r_{j_k}}(B_0)) \to L$.
- Every convergent subsequence of the bounded sequence $\{L_j\}$ has limit L. Hence, $L_j \to L$.
- μ has strong derivative L at 0.

Step 3:

- Set $\mu_k(E) = r_{j_k}^{-Q}\mu(\delta_{r_{j_k}}(E))$. Then $u_{j_k} = \Gamma[\mu_k] \to L = \Gamma[Lm]$ normally.
- $\mu_{r_{j_k}}(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.
- Put $B = B_0$ to conclude that $L_{j_k} = \mu(\delta_{r_{j_k}}(B_0))/m(\delta_{r_{j_k}}(B_0)) \to L$.
- Every convergent subsequence of the bounded sequence $\{L_j\}$ has limit L. Hence, $L_j \to L$.
- μ has strong derivative L at 0.

Step 3:

- Set $\mu_k(E) = r_{j_k}^{-Q}\mu(\delta_{r_{j_k}}(E))$. Then $u_{j_k} = \Gamma[\mu_k] \to L = \Gamma[Lm]$ normally.
- $\mu_{r_{j_k}}(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.
- Put $B = B_0$ to conclude that $L_{j_k} = \mu(\delta_{r_{j_k}}(B_0))/m(\delta_{r_{j_k}}(B_0)) \to L$.
- Every convergent subsequence of the bounded sequence $\{L_j\}$ has limit L. Hence, $L_j \to L$.
- μ has strong derivative L at 0.

Step 3:

- Set $\mu_k(E) = r_{j_k}^{-Q}\mu(\delta_{r_{j_k}}(E))$. Then $u_{j_k} = \Gamma[\mu_k] \to L = \Gamma[Lm]$ normally.
- $\mu_{r_{j_k}}(B) \to Lm(B)$ for every $d_{\mathcal{L}}$-ball B.
- Put $B = B_0$ to conclude that $L_{j_k} = \mu(\delta_{r_{j_k}}(B_0))/m(\delta_{r_{j_k}}(B_0)) \to L$.
- Every convergent subsequence of the bounded sequence $\{L_j\}$ has limit L. Hence, $L_j \to L$.
- μ has strong derivative L at 0.

Remark

Work in progress to see whether this theorem can be proved for more general class of measures satisfying Brossard-Chevalier type condition:

$$\sup_{(x,t)\in B\times(0,t_0)} \left(\Gamma[|\mu|](x,t) - |\Gamma[\mu](x,t)|\right) < \infty,$$

where B is a $d_{\mathcal{L}}$-ball around 0, and $t_0 > 0$.

Remark

Work in progress to see whether this theorem can be proved for more general class of measures satisfying Brossard-Chevalier type condition:

$$\sup_{(x,t)\in B\times(0,t_0)} \left(\Gamma[|\mu|](x,t) - |\Gamma[\mu](x,t)|\right) < \infty,$$

where B is a $d_{\mathcal{L}}$-ball around 0, and $t_0 > 0$.

References I

Christian Bär.
Some properties of solutions to weakly hypoelliptic equations.
Int. J. Differ. Equ., pages Art. ID 526390, 8, 2013.
doi:10.1155/2013/526390.

Jean Brossard and Lucien Chevalier.
Problème de Fatou ponctuel et dérivabilité des mesures.
Acta Math., 164(3-4):237–263, 1990.
doi:10.1007/BF02392755.

A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni.
Uniform Gaussian estimates for the fundamental solutions for heat operators on Carnot groups.
Adv. Differential Equations, 7(10):1153–1192, 2002.

References II

Andrea Bonfiglioli and Francesco Uguzzoni.
Representation formulas and Fatou-Kato theorems for heat operators on stratified groups.
Rend. Mat. Appl. (7), 25(1):53–67, 2005.

F. W. Gehring.
The boundary behavior and uniqueness of solutions of the heat equation.
Trans. Amer. Math. Soc., 94:337–364, 1960.
doi:10.2307/1993430.

Lynn H. Loomis.
The converse of the Fatou theorem for positive harmonic functions.
Trans. Amer. Math. Soc., 53:239–250, 1943.
doi:10.2307/1990345.

References III

Wade Ramey and David Ullrich.
On the behavior of harmonic functions near a boundary point.
Trans. Amer. Math. Soc., 305(1):207–220, 1988.
doi:10.2307/2001049.

Jayanta Sarkar.
Boundary behavior of positive solutions of the heat equation on a stratified Lie group.
2021.
arXiv:2012.11234.

Jayanta Sarkar.
On parabolic convergence of positive solutions of the heat equation.
Complex Variables and Elliptic Equations, 0(0):1–17, 2021.
doi:10.1080/17476933.2021.1882432.

Thank you!