Weighted estimates for maximal product of spherical averages

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- $\langle f \rangle_Q := \frac{1}{|Q|} \int_Q |f(y)| dy$ and $\langle f \rangle_{Q,p} := \left(\frac{1}{|Q|} \int_Q |f(y)|^p dy\right)^{\frac{1}{p}}$.

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- For any locally integrable function f and a point $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function M is defined as $Mf(x) := \sup_{Q \ni x} \langle f \rangle_Q$.

Spherical averages and maximal functions

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$$\mathcal{A}_r f(x) := \int_{\mathbb{S}^{n-1}} f(x - ry) d\sigma(y).$$

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$$M_{lac} f(x) := \sup_{j\in\mathbb{Z}} |\mathcal{A}_{2^j} f(x)|$$

L^p estimates of M_{full} and M_{lac}

Theorem (Stein; Proc. Nat. Acad. Sci., vol 73, 1976)

Let $n \geq 3$. Then the operator M_{full} maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$.

Later J. Bourgain [On the spherical maximal function in the plane, IHES 1985]extended the above result to dimension $n = 2$, i.e M_{full} maps $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ if and only if $p > 2$.

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Theorem (Calderón; illinois J. of Math. vol 23, no. 3, 1979 Coifman and Weiss; Bull. Amer. Math. Soc. 84(1978))

Let $n \geq 2$. Then the operator M_{lac} maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p > 1$.

Boundedness of M_{full} and M_{lac} w.r.t power weights

Theorem (J. Duoandikoetxea, L. Vega; J. London Math. Soc. (2) 53 (1996))

Let $n \geq 2$. Then

- M_{full} *is bounded on* $L^p(|x|^\alpha)$ *for* $p > \frac{n}{n-1}$ *and* $1 - n < \alpha < (n-1)(p-1) - 1$. *The range of* α *is sharp except possibly at the point* $\alpha = 1 - n$.
- M_{lac} *is bounded on* $L^p(|x|^\alpha)$ *if and only if* $1 - n \leq \alpha < (n-1)(p-1)$ *for* $1 < p < \infty$.

Sparse family and sparse forms

Definition

A collection of cubes $\mathcal{S}$ in $\mathbb{R}^n$ is said to be η- sparse $(0 < \eta < 1)$ if there are sets $\{E_Q \subset Q : Q \in \mathcal{S}\}$ which are pairwise disjoint and satisfy $|E_Q| \geq \eta |Q|$ for all $Q \in \mathcal{S}$.

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Example: Let $\mathcal{S} = \{Q^k \subset \mathbb{R}^n : Q^k = \prod_{j=1}^{n}[0, 2^{k_j}); k_j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n)\}$ and $E_{Q^k} = \prod_{j=1}^{n}[2^{k_j - 1}, 2^{k_j})$. Observe that $|E_{Q^k}| \geq \eta|Q^k|$ with $\eta \leq 2^{-n}$. Therefore, $\mathcal{S}$ is a sparse family.

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Let $1 \leq r, s < \infty$. Then for any compactly supported bounded functions f, g we define the bilinear sparse form as

$$\Lambda_{\mathcal{S},r,s}(f,g) := \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{Q,r} \langle g \rangle_{Q,s}$$

The trilinear (p, q, r)-sparse form is defined as

$$\Lambda_{\mathcal{S},p,q,r}(f,g,h) = \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{Q,p} \langle g \rangle_{Q,q} \langle h \rangle_{Q,r}.$$

M. Lacey's result on M_{lac} and M_{full}

Theorem (M. Lacey; J. D'Analyse Mathématique, vol 139(2019))

Let $n \geq 2$. Then for any compactly supported bounded functions f, g and $(\frac{1}{r}, \frac{1}{s})$ in the interior of L_n (respectively F_n), the operator M_{lac} (respectively M_{full}) satisfies the following inequality

$$\langle Tf, g \rangle \lesssim \Lambda_{\mathcal{S},r,s}(f, g),$$

where $T = M_{\text{lac}}$ (respectively M_{full}).

The triangle L_n

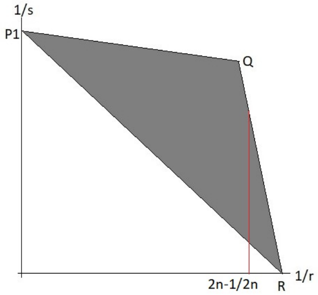

Figure: L_n

where the points are $P_1 = (0,1), R = (1,0)$ and $Q = (\frac{n}{n+1}, \frac{n}{n+1})$.

The trapezium F_n

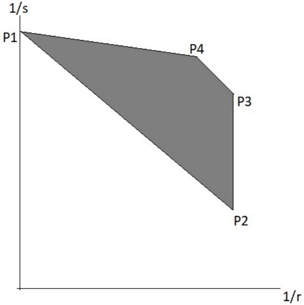

Figure: F_n, where $P_1 = (0, 1)$, $P_2 = (\frac{n-1}{n}, \frac{1}{n})$, $P_3 = (\frac{n-1}{n}, \frac{n-1}{n})$ and $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$

Maximal product of spherical averages

The operator maximal product of spherical averages is defined by

$$\mathcal{M}_{\text{full}}(f_1, f_2)(x) := \sup_{r>0} |\mathcal{A}_r f_1(x) \mathcal{A}_r f_2(x)|.$$

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Similarly, dyadic maximal product of spherical averages is defined by

$$\mathcal{M}_{\text{lac}}(f_1, f_2)(x) := \sup_{j\in\mathbb{Z}} |\mathcal{A}_{2^j} f_1(x)\mathcal{A}_{2^j} f_2(x)|.$$

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The bilinear spherical maximal operator is defined by

$$\mathcal{M}_{sph}(f_1, f_2)(x) := \sup_{r>0} \int_{\mathbb{S}^{2n-1}} |f_1(x - ry) f_2(x - rz)| d\sigma_{2n-1}(y, z)$$

Boundedness of bilinear spherical maximal function

Lemma (E. Jeong, S. Lee; J. Funct. Anal. 2020)

Let $n \geq 2$. Then

$$\mathcal{M}_{\mathrm{sph}}(f,g)(x) \lesssim Mf(x) M_{\mathrm{full}} g(x)$$
and
$$\mathcal{M}_{\mathrm{sph}}(f,g)(x) \lesssim M_{\mathrm{full}} f(x) Mg(x),$$

where M is the Hardy-Littlewood maximal function.

Theorem (E. Jeong, S. Lee; J. Funct. Anal. 2020)

Let $n \geq 2$. Let $1 \leq p, q \leq \infty$ and $0 < r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then the following inequality

$$\|\mathcal{M}_{\mathrm{sph}}(f, g)\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{1}$$

holds if and only if $r > \frac{n}{2n-1}$ except the case $(p, q, r) = (1, \infty, 1)$ or $(\infty, 1, 1)$.

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holds if and only if $r > \frac{n}{2n-1}$ except the case $(p,q,r) = (1,\infty,1)$ or $(\infty,1,1)$. In addition, the weak type estimates holds in terms of Lorentz spaces. i.e.

$$\|\mathcal{M}_{\text{sph}}(f,g)\|_{L^{r,u}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,s}(\mathbb{R}^n)} \|g\|_{L^{q,t}(\mathbb{R}^n)} \tag{2}$$

holds in the following cases

- *If $p = r = 1$ with $u = t = \infty$ and $s = 1$.*
- *For $n \geq 3$, if $p = 1$, $q = \frac{n}{n-1}$ then (2) holds with $u = \infty$ and $s = t = 1$.*
- *For $n \geq 3$, if $1 < p < \frac{n}{n-1}$, $r = \frac{n}{2n-1}$, then (2) holds with $u = \infty$ and s, t satisfy $\frac{1}{s} + \frac{1}{t} = \frac{2n-1}{n}$ and $s, t > 0$.*

Bilinear $A_{\vec{p},\vec{r}}$ weights

Definition (K.Li, J.M. Martell, S. Ombrosi; Adv. in Math., 2020)

Let $\vec{p} = (p_1, p_2)$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 \leq p_1, p_2 < \infty$. For a tuple $\vec{r} = (r_1, r_2, r_3)$ with $r_i \leq p_i$, $i = 1, 2$, and $r_3' > p$, where $1 \leq r_1, r_2, r_3 < \infty$, we say that $\vec{w} = (w_1, w_2) \in A_{\vec{p},\vec{r}}$ if $0 < w_i < \infty$ a.e. for $i = 1, 2$ and

$$[\vec{w}]_{A_{\vec{p},\vec{r}}} := \sup_{Q \subset \mathbb{R}^n} \langle v_w^{\frac{r_3'}{r_3'-p}} \rangle_Q^{\frac{1}{p} - \frac{1}{r_3'}} \prod_{i=1}^{2} \langle w_i^{\frac{r_i}{r_i - p_i}} \rangle_Q^{\frac{1}{r_i} - \frac{1}{p_i}} < \infty,$$

where $v_w := \prod_{i=1}^2 w_i^{p/p_i}$. When $r_3 = 1$, the term corresponding to v_w needs to be replaced by $\langle v_w \rangle_Q^{1/p}$. Analogously, when $p_i = r_i$, the term corresponding to w_i needs to be replaced by $ess \sup_Q w_i^{-1/p_i}$.

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$$[\vec{w}]_{A_{\vec{p},\vec{r}}} := \sup_{Q \subset \mathbb{R}^n} \langle v_w^{\frac{r_3'}{r_3'-p}} \rangle_Q^{\frac{1}{p} - \frac{1}{r_3'}} \prod_{i=1}^{2} \langle w_i^{\frac{r_i}{r_i - p_i}} \rangle_Q^{\frac{1}{r_i} - \frac{1}{p_i}} < \infty,$$

where $v_w := \prod_{i=1}^{2} w_i^{p/p_i}$. When $r_3 = 1$, the term corresponding to v_w needs to be replaced by $\langle v_w \rangle_Q^{1/p}$. Analogously, when $p_i = r_i$, the term corresponding to w_i needs to be replaced by $ess\sup_Q w_i^{-1/p_i}$.

When $\vec{r} = (1, 1, 1)$, the weight class $A_{\vec{p},\vec{r}}$ coinsides with $A_{\vec{p}}$, which was introduced by Lerner et al. Adv. in Math., 2009

Theorem 1: Weighted boundedness of $\mathcal{M}_{\text{lac}}$ and $\mathcal{M}_{\text{full}}$

Theorem (-, S. Shrivastava, L. Roncal; J. Fourier Anal. Appl. 2021)

Let $n \geq 2$. For $i = 1, 2$, let $(\frac{1}{r_i}, \frac{1}{s_i})$ be in the interior of L_n (respectively F_n). Assume that $t = \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2} > 1$. Then for all $\vec{q} = (q_1, q_2)$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ with $r_i < q_i$, $i = 1, 2$, and $t' > q$, the operator $\mathcal{M}_{\text{lac}}$ (respectively $\mathcal{M}_{\text{full}}$) extends to a bounded operator from $L^{q_1}(w_1) \times L^{q_2}(w_2) \to L^q(v_w)$, i.e.,

$$\|\mathcal{M}(f_1, f_2)\|_{L^q(v_w)} \leq C([\vec{w}]_{A_{\vec{q},\vec{r}}}) \prod_{i=1}^{2} \|f_i\|_{L^{q_i}(w_i)},$$

where $\mathcal{M} := \mathcal{M}_{\text{lac}}$ (respectively $\mathcal{M}_{\text{full}}$) and $\vec{w} = (w_1, w_2) \in A_{\vec{q},\vec{r}}$ with $\vec{r} = (r_1, r_2, t)$.

Theorem 2: Boundedness of $\mathcal{M}_{\text{lac}}$ w.r.t power (or radial) weights

Theorem (-, S. Shrivastava, L. Roncal; J. Fourier Anal. Appl. 2021)

Let $n \geq 2$. The operator $\mathcal{M}_{\text{lac}}$ is bounded from $L^p(|x|^\alpha) \times L^p(|x|^\beta)$ to $L^{p/2}(|x|^{\frac{\alpha+\beta}{2}})$ with $1 < p \leq \frac{2n}{2n-1}$ for α, β satisfying:

$$(1-n)p < \alpha, \beta < (n-1)(p-1) \text{ and } \alpha + \beta > 2(1-n)(n-(n-1)p).$$

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Define

- $\mathcal{R}_p = \{\omega_a(x) = |x|^a : 1 - n \leq a < (n-1)(p-1)\}.$

Sparse domination of $\mathcal{M}_{\text{lac}}$ and $\mathcal{M}_{\text{full}}$

Theorem (-,S. Shrivastava, L. Roncal; J. Fourier Anal. Appl. 2021)

Let $n \geq 2$. For $i = 1, 2$, let $(\frac{1}{r_i}, \frac{1}{s_i})$ be in the interior of L_n (respectively F_n) and $\rho_i > r_i$. Then for any non-negative compactly supported bounded functions f_1, f_2 and h, there exists a sparse collection $\mathcal{S} = \mathcal{S}_{\rho_1,\rho_2,t}$ such that

$$\langle \mathcal{M}(f_1, f_2), h \rangle \leq C\Lambda_{\mathcal{S}_{\rho_1,\rho_2,t}}(f_1, f_2, h),$$

where $t := \frac{s_1 s_2}{s_1+s_2-s_1 s_2} > 1$ and $\mathcal{M} := \mathcal{M}_{\text{lac}}$ (respectively $\mathcal{M}_{\text{full}}$).

Sharpness of sparse domination

Sharpness of sparse bound for $\mathcal{M}_{\text{lac}}$:

- $\frac{1}{r_1} + \frac{1}{r_2} + \frac{n}{t} \leq n$
- $\frac{1}{r_1} + \frac{n}{s_1} + \frac{1}{r_2} + \frac{n}{s_2} \leq 2n$
- $\frac{n}{r_1} + \frac{1}{s_1} + \frac{n}{r_2} + \frac{1}{s_2} \leq 2n.$

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Sharpness of sparse bound for $\mathcal{M}_{\text{full}}$:

- $r_1, r_2 > \frac{n}{n-1}$
- $\frac{1}{r_1} + \frac{n}{s_1} + \frac{1}{r_2} + \frac{n}{s_2} \leq 2n$
- $\frac{n+1}{r_1} + \frac{n-1}{s_1} + \frac{n+1}{r_2} + \frac{n-1}{s_2} \leq 4(n-1).$

Proof of Theorem 2

Step I: Let $1 < \tilde{p}_1 = \tilde{p}_2 \leq \frac{2n}{2n-1}$. Now consider $\frac{2n}{2n-1} < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $(\frac{1}{r_i}, \frac{1}{s_i}) \in L_n$, $i = 1, 2$ with $t = \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2} > 1$. For $\vec{r} = (r_1, r_2, t) < \vec{p} := (p_1, p_2, p)$, let $\vec{w} = (w_1, w_2) \in A_{\vec{p},\vec{r}}$. By Theorem 1 we have

$$\|\mathcal{M}_{\text{lac}}(f_1, f_2)\|_{L^p(w)} \leq C_1 \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}. \tag{3}$$

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Also, note that using the product type estimates we get,

$$\|\mathcal{M}_{\mathrm{lac}}(f_1, f_2)\|_{L^q(v)} \leq C_2 \|f_1\|_{L^{q_1}(v_1)} \|f_2\|_{L^{q_2}(v_2)}, \tag{4}$$

for $1 < q_i < \tilde{p}_i$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $v_i \in \left(A_{\frac{q_i}{t_i}} \cap \mathrm{RH}_{\left(\frac{\phi'_{\mathrm{lac}}(\frac{1}{t_i})}{q_i}\right)'}\right) \cup \mathcal{R}_{q_i}$,

$v = v_1^{\frac{q}{q_1}} v_2^{\frac{q}{q_2}}$ and $(\frac{1}{t_i}, \frac{1}{\eta_i}) \in L_n$ for some $\eta_i \in (1, \infty)$ and $1 < t_i < q_i < \eta_i'$, for $i = 1, 2$.

Proof continued...

We consider the linearised operator $\mathcal{M}_{\text{lac}}$ as follows

$$\mathcal{M}_{\text{lac}}(f_1, f_2)(x) = \mathcal{A}_{\tau(x)} f_1(x) \mathcal{A}_{\tau(x)} f_2(x),$$

where τ is a measurable function from $\mathbb{R}^n$ to $[0, \infty)$.

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where τ is a measurable function from $\mathbb{R}^n$ to $[0, \infty)$. For $z \in S := \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}$, consider the functions

$$\frac{1}{l(z)} := \frac{1-z}{p} + \frac{z}{q}, \qquad \frac{1}{l_i(z)} := \frac{1-z}{p_i} + \frac{z}{q_i}, \qquad i = 1, 2.$$

Choose $\theta \in (0, 1)$ such that

$$\frac{1}{l(\theta)} := \frac{1-\theta}{p} + \frac{\theta}{q} = \frac{1}{\tilde{p}}, \qquad \frac{1}{l_i(\theta)} := \frac{1-\theta}{p_i} + \frac{\theta}{q_i} = \frac{1}{\tilde{p}_i}, \qquad i = 1, 2.$$

Proof continued...

Note that for any linear operator T and a positive number $k \in (0,1)$ satisfying $\frac{k}{p} + \frac{k}{q} < 1$ and $k < \tilde{p}$, we can write the following

$$\|Tf\|_{L^{\tilde{p}}}^{k} = \||Tf|^{k}\|_{L^{\frac{\tilde{p}}{k}}} = \sup_{\substack{g \in L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n) \\ \|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)}=1}} \Big| \int_{\mathbb{R}^n} |Tf|^{k} g \Big|.$$

Proof continued...

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$$\|Tf\|_{L^{\tilde{p}}}^k = \||Tf|^k\|_{L^{\frac{\tilde{p}}{k}}} = \sup_{\substack{g \in L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n) \\ \|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1}} \Big| \int_{\mathbb{R}^n} |Tf|^k g \Big|.$$

Consider

$$\widetilde{v}_N(x) = v(x), \quad \text{if} \quad v(x) \leq N \quad \text{and} \quad \widetilde{v}_N(x) = N, \quad \text{if} \quad v(x) > N,$$
$$\widetilde{w}_N(x) = w(x), \quad \text{if} \quad w(x) \leq N \quad \text{and } \widetilde{w}_N(x) = N, \quad \text{if} \quad w(x) > N.$$

Let f_1, f_2 be finite simple functions and g be a non-negative finite simple function such that $\|f_i\|_{L^{\tilde{p}_1}(\mathbb{R}^n)} = 1$, for $i = 1, 2$ and $\|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1$.

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Now consider the following function

$$\psi(z) := \int_{\mathbb{R}^n} \left| \mathcal{A}_{\tau(x)} f_{1,z}(x) \mathcal{A}_{\tau(x)} f_{2,z}(x) \widetilde{v}_N^{\frac{z}{q}} \widetilde{w}_N^{\frac{1-z}{p}} g^{\frac{(1-\frac{k}{l(z)})}{k(1-\frac{k}{\tilde{p}})}} \right|^k dx, \quad (5)$$

where

$$f_{j,z}(x) := |f_j(x)|^{\frac{\tilde{p}_j}{l_j(z)}} e^{iu_j} (v_j + \epsilon)^{\frac{-z}{q_j}} (w_j + \epsilon)^{\frac{z-1}{p_j}}, \quad j = 1, 2,$$

for $z \in S$, $\epsilon > 0$ and $u_j \in [0, 2\pi]$.

Note that we have the following expression for $\psi(\theta)$, $\theta \in (0,1)$,

$$\psi(\theta) = \int_{\mathbb{R}^n} \Big| \prod_{j=1}^{2} \mathcal{A}_{\tau(x)}(f_j(v_j+\epsilon)^{-\frac{\theta}{q_j}}(w_j+\epsilon)^{\frac{\theta-1}{p_j}})(x)\widetilde{v}_N^{\frac{\theta}{q}}\widetilde{w}_N^{\frac{1-\theta}{p}}\Big|^k g(x)dx.$$

For each $x \in \mathbb{R}^n$, the functions $\mathcal{A}_{\tau(x)}f_{j,z}(x)$, $\widetilde{v}_N^{\frac{z}{q}}(x)$, $\widetilde{w}_N^{\frac{1-z}{p}}(x)$ and $g^{\frac{(1-\frac{k}{l(z)})}{k(1-\frac{k}{\tilde{p}})}}(x)$ are analytic in the domain $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$. Also, using the Hölder's inequality with exponents $\frac{p}{k}$ and $\frac{p}{p-k}$, it is easy to see that ψ is a bounded function.

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For each $x \in \mathbb{R}^n$, the functions $\mathcal{A}_{\tau(x)} f_{j,z}(x)$, $\widetilde{v}_N^{\frac{z}{q}}(x)$, $\widetilde{w}_N^{\frac{1-z}{p}}(x)$ and $g^{\frac{(1-\frac{k}{l(z)})}{k(1-\frac{k}{\tilde{p}})}}(x)$ are analytic in the domain $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$.
Also, using the Hölder's inequality with exponents $\frac{p}{k}$ and $\frac{p}{p-k}$, it is easy to see that ψ is a bounded function.
Moreover, the Hölder's inequality with exponents $\frac{p}{k}$ and $\frac{p}{p-k}$ and the fact that $\|f_i\|_{L^{\tilde{p}_i}(\mathbb{R}^n)} = 1$, $i = 1, 2$ and $\|g\|_{L^{\frac{\tilde{p}}{\tilde{p}-k}}(\mathbb{R}^n)} = 1$, yield that

$$|\psi(it)| \leq C_1^k.$$

Similarly, using the Hölder's inequality with exponents $\frac{q}{k}$ and $\frac{q}{q-k}$, we get

$$|\psi(1+it)| \leq C_2^k.$$

We invoke the maximum modulus principle for subharmonic functions to deduce that

$$|\psi(\theta)| = \int_{\mathbb{R}^n} \Big| \prod_{j=1}^{2} \mathcal{A}_{\tau(x)}\Big(f_j v_{j,\epsilon}^{-\frac{\theta}{q_j}} w_{j,\epsilon}^{\frac{\theta-1}{p_j}}\Big)(x) \widetilde{v}_N^{\frac{\theta}{q}} \widetilde{w}_N^{\frac{1-\theta}{p}} \Big|^k g(x)dx \leq C_1^{k(1-\theta)} C_2^{k\theta}.$$

Here we have used the notation $v_{j,\epsilon} = v_j + \epsilon$ and $w_{j,\epsilon} = w_j + \epsilon$ for $j = 1, 2$.

We invoke the maximum modulus principle for subharmonic functions to deduce that

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Here we have used the notation $v_{j,\epsilon} = v_j + \epsilon$ and $w_{j,\epsilon} = w_j + \epsilon$ for $j = 1, 2$. Therefore, using a duality argument we obtain that

$$\Big(\int_{\mathbb{R}^n} \Big(\Big| \mathcal{A}_{\tau(x)}(f_1 v_{1,\epsilon}^{-\frac{\theta}{q_1}} w_{1,\epsilon}^{\frac{\theta-1}{p_1}})(x) \mathcal{A}_{\tau(x)}(f_2 v_{2,\epsilon}^{-\frac{\theta}{q_2}} w_{2,\epsilon}^{\frac{\theta-1}{p_2}})(x) \Big| \widetilde{v}_N^{\frac{\theta}{q}} \widetilde{w}_N^{\frac{1-\theta}{p}} \Big)^{\tilde{p}} dx \Big)^{\frac{1}{\tilde{p}}}$$
$$\leq C \Big(\int_{\mathbb{R}^n} |f_1|^{\tilde{p}_1} \Big)^{\frac{1}{\tilde{p}_1}} \Big(\int_{\mathbb{R}^n} |f_2|^{\tilde{p}_2} \Big)^{\frac{1}{\tilde{p}_2}}.$$

Since the set of finite simple functions is dense in $L^s(\mathbb{R}^n),\ 1 \leq s < \infty$, we get the above estimate for all $L^{\tilde{p}_1}(\mathbb{R}^n)$ functions f_1 and f_2 (note that we have assumed $\tilde{p}_1 = \tilde{p}_2$).

Since the set of finite simple functions is dense in $L^s(\mathbb{R}^n),\ 1 \leq s < \infty$, we get the above estimate for all $L^{\tilde{p}_1}(\mathbb{R}^n)$ functions f_1 and f_2 (note that we have assumed $\tilde{p}_1 = \tilde{p}_2$).Next, recall that the constants C_1, C_2 are independent of ϵ, N and τ. Let $\epsilon \to 0$ and $N \to \infty$ and replace f_i by $f_i v_i^{\frac{\theta}{q_i}} w_i^{\frac{1-\theta}{p_i}}$, $i = 1, 2$, in the above to get that

$$\left(\int_{\mathbb{R}^n} \left(\left| \mathcal{A}_{\tau(x)}(f_1)(x) \mathcal{A}_{\tau(x)}(f_2)(x) \right| v^{\frac{\theta}{q}} w^{\frac{1-\theta}{p}} \right)^{\tilde{p}} dx \right)^{\frac{1}{\tilde{p}}}$$
$$\leq C \left(\int_{\mathbb{R}^n} |f_1|^{\tilde{p}_1} (v_1^{\frac{\theta}{q_1}} w_1^{\frac{1-\theta}{p_1}})^{\tilde{p}_1} \right)^{\frac{1}{\tilde{p}_1}} \left(\int_{\mathbb{R}^n} |f_2|^{\tilde{p}_2} (v_2^{\frac{\theta}{q_2}} w_2^{\frac{1-\theta}{p_2}})^{\tilde{p}_2} \right)^{\frac{1}{\tilde{p}_2}}.$$

Since the set of finite simple functions is dense in $L^s(\mathbb{R}^n)$, $1 \leq s < \infty$, we get the above estimate for all $L^{\tilde{p}_1}(\mathbb{R}^n)$ functions f_1 and f_2 (note that we have assumed $\tilde{p}_1 = \tilde{p}_2$).Next, recall that the constants C_1, C_2 are independent of ϵ, N and τ. Let $\epsilon \to 0$ and $N \to \infty$ and replace f_i by $f_i v_i^{\frac{\theta}{q_i}} w_i^{\frac{1-\theta}{p_i}}$, $i = 1, 2$, in the above to get that

$$\left(\int_{\mathbb{R}^n} \left(\left| \mathcal{A}_{\tau(x)}(f_1)(x) \mathcal{A}_{\tau(x)}(f_2)(x) \right| v^{\frac{\theta}{q}} w^{\frac{1-\theta}{p}} \right)^{\tilde{p}} dx \right)^{\frac{1}{\tilde{p}}} \leq C \left(\int_{\mathbb{R}^n} |f_1|^{\tilde{p}_1} (v_1^{\frac{\theta}{q_1}} w_1^{\frac{1-\theta}{p_1}})^{\tilde{p}_1} \right)^{\frac{1}{\tilde{p}_1}} \left(\int_{\mathbb{R}^n} |f_2|^{\tilde{p}_2} (v_2^{\frac{\theta}{q_2}} w_2^{\frac{1-\theta}{p_2}})^{\tilde{p}_2} \right)^{\frac{1}{\tilde{p}_2}}.$$

Since the constant C is independent of τ, therefore we get the boundedness of the operator $\mathcal{M}_{\text{lac}}$, i.e.

$$\left(\int_{\mathbb{R}^n} \left(\left| \mathcal{M}_{\text{lac}}(f_1, f_2)(x) \right| v^{\frac{\theta}{q}} w^{\frac{1-\theta}{p}} \right)^{\tilde{p}} dx \right)^{\frac{1}{\tilde{p}}} \leq C \left(\int_{\mathbb{R}^n} |f_1|^{\tilde{p}_1} (v_1^{\frac{\theta}{q_1}} w_1^{\frac{1-\theta}{p_1}})^{\tilde{p}_1} \right)^{\frac{1}{\tilde{p}_1}} \left(\int_{\mathbb{R}^n} |f_2|^{\tilde{p}_2} (v_2^{\frac{\theta}{q_2}} w_2^{\frac{1-\theta}{p_2}})^{\tilde{p}_2} \right)^{\frac{1}{\tilde{p}_2}}. \quad (6)$$

Step II: For $\epsilon > 0$, let $p_1 = p_2 = \frac{2n}{2n-1} + 2\epsilon$, $r_1 = r_2 = \frac{2n}{2n-1} + \epsilon$ and $(\frac{1}{r_i}, \frac{1}{s_i}) \in L_n$. Check that for this choice of $(\frac{1}{r_i}, \frac{1}{s_i})$, $t = \frac{s_1 s_2}{s_1 + s_2 - s_1 s_2} > 1$ and set $\vec{r} = (r_1, r_2, r_3)$ with $r_3 = t$. Let $\vec{w} = (|x|^{\alpha'}, |x|^{\beta'}) \in A_{\vec{p},\vec{r}}$ and note that the estimate (3) holds for bilinear $A_{\vec{p},\vec{r}}$ weights. Next, for $0 < \delta < \tilde{p}_1 - 1$, choose $q_1 = q_2 = \tilde{p}_1 - \delta$ and $\vec{v} = (|x|^a, |x|^b)$ with $1 - n \leq a, b < (n-1)(\tilde{p}_1 - \delta - 1)$. Then we know that the estimate (4) holds for $\mathcal{M}_{\mathrm{lac}}$.

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$$\Big(\int \Big|\mathcal{M}_{\text{lac}}(f_1, f_2)\Big|^{\tilde{p}} (|x|^{\frac{(a+b)\theta}{\tilde{p}_1-\delta} + \frac{(\alpha'+\beta')(1-\theta)(2n-1)}{2(n+\epsilon(2n-1))}})^{\tilde{p}} \Big)^{\frac{1}{\tilde{p}}}$$

$$\lesssim \Big(\int (|f_1||x|^{\frac{a\theta}{\tilde{p}_1-\delta} + \frac{\alpha'(1-\theta)(2n-1)}{2n+2\epsilon(2n-1)}})^{\tilde{p}_1} \Big)^{\frac{1}{\tilde{p}_1}} \Big(\int (|f_2||x|^{\frac{b\theta}{\tilde{p}_2-\delta} + \frac{\beta'(1-\theta)(2n-1)}{2n+2\epsilon(2n-1)}})^{\tilde{p}_2} \Big)^{\frac{1}{\tilde{p}_2}} \tag{7}$$

Now, we show that the exponents of weights in the estimate above may be chosen suitably so that they satisfy the following conditions, i.e. $\vec{w} = (|x|^{\alpha'}, |x|^{\beta'}) \in A_{\vec{p},\vec{r}}$. This implies that

$$|x|^{\frac{\alpha'\theta_1(2n-1)}{2n+2\epsilon(2n-1)}} \in A_{(\frac{1-r}{r})\theta_1}, \quad |x|^{\frac{\beta'\theta_2(2n-1)}{2n+2\epsilon(2n-1)}} \in A_{(\frac{1-r}{r})\theta_2},$$

$$\text{and} \qquad |x|^{\frac{(\alpha'+\beta')\delta_3(2n-1)}{2n+2\epsilon(2n-1)}} \in A_{\frac{1-r}{r}\delta_3},$$

where

$$\frac{1}{\delta_i} = \frac{1}{r_i} - \frac{1}{p_i}, \quad \frac{1}{\theta_i} = \frac{1-r}{r} - \frac{1}{\delta_i} \quad \text{and} \quad p_3 = p', \quad i = 1, 2, 3.$$

Substituting the values of the various parameters, we obtain

$$\frac{1-r}{r} = \frac{(2n-2)(1+\epsilon(2n-1))}{2n+\epsilon(2n-1)},$$
$$\frac{1}{\theta_i} = \frac{\epsilon(2n-1)(2n-2)-1}{2n+\epsilon(2n-1)} + \frac{2n-1}{2n+2\epsilon(2n-1)}, \quad \text{for} \quad i=1,2,$$
$$\frac{1}{\delta_3} = \frac{\epsilon(2n-1)^2}{2n+\epsilon(2n-1)} + \frac{2n-1}{n+\epsilon(2n-1)} - 1.$$

Substituting the values of the various parameters, we obtain

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$$\frac{1}{\delta_3} = \frac{\epsilon(2n-1)^2}{2n+\epsilon(2n-1)} + \frac{2n-1}{n+\epsilon(2n-1)} - 1.$$

Since ϵ can be chosen arbitrarily small, therefore taking $\epsilon \to 0$ we get $(1-n)\frac{2n}{2n-1} < \alpha', \beta' < 0$ and $(1-n)\frac{2n}{2n-1} < \alpha' + \beta' < 0$. Now taking $\delta \to \tilde{p}_1 - 1$, we get $\theta = \frac{2n}{\tilde{p}_1} - (2n-1)$. Since the range of α' and β' is an open set, we get that $\mathcal{M}_{\text{lac}}$ is bounded from $L^{\tilde{p}_1}(|x|^{\alpha}) \times L^{\tilde{p}_2}(|x|^{\beta})$ to $L^{\tilde{p}}(|x|^{\frac{\alpha+\beta}{2}})$ for α, β satisfying

$$(1-n)\tilde{p}_1 < \alpha, \beta < 0 \quad \text{and} \quad \alpha+\beta > 2(1-n)\big(n-(n-1)\tilde{p}_1\big).$$

Further, using the product-type weighted boundedness of $\mathcal{M}_{\text{lac}}$ for $\tilde{p}_1 = \tilde{p}_2$, we get $\mathcal{M}_{\text{lac}}$ is bounded from $L^{\tilde{p}_1}(|x|^a) \times L^{\tilde{p}_2}(|x|^b)$ to $L^{\tilde{p}}(|x|^{\frac{a+b}{2}})$ for $1 - n \leq a, b < (n-1)(\tilde{p}_1 - 1)$.
This proves the desired result for the operator $\mathcal{M}_{\text{lac}}$.

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THANK YOU!