Sharp Adams type inequalities for the fractional Laplace-Beltrami operator on noncompact symmetric spaces

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Sobolev Inequalities

Let $1 \leq p < n$. Then $\exists\ S = S_{n,p} > 0$ such that

$$S \left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{p/p^*} \leq \int_{\mathbb{R}^n} |\nabla u|^p,$$

holds for all $u \in C_c^1(\mathbb{R}^n)$. Here, $p^* = \frac{np}{n-p}$.

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There exists $C > 0$ such that

$$\sup_{u \in C_c^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |\nabla u|^p \leq 1} \int_{\mathbb{R}^n} |u|^{p^*} \leq C.$$

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For $p = n$, the inequality fails. Counterexample: let

$$f(x) = \log\log(1 + \frac{1}{|x|}), \ \ x \in B(0,1).$$

Then $f \in W^{1,n}(\mathbb{R}^n)$ but $f \notin L^\infty(\mathbb{R}^n)$.

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For $p \in [1, \infty]$, $W^{1,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \nabla u \in L^p(\mathbb{R}^n)\}$.

Moser-Trudinger Inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $\exists\ C_n > 0$ such that

$$\sup_{u \in C_c^1(\Omega), \int_\Omega |\nabla u|^n \le 1} \int_\Omega e^{\alpha |u(x)|^{n/(n-1)}}\ dx \le C_n |\Omega|,$$

holds for every $\alpha \le \alpha_n = n[\omega_{n-1}]^{1/(n-1)}$, where ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$.

Furthermore, the constant α_n is sharp.

Theorem (Adams, 1988)

Let $\Omega \subset \mathbb{R}^n$ be bounded domain $m \in \mathbb{N}$ with $m < n$ and $p = n/m$. Then $\exists\ c_0 = c_0(n, m) > 0$ such that

$$\sup_{u \in C_c^\infty(\Omega), \|\nabla^m u\|_p \le 1} \int_\Omega e^{\beta |u(x)|^{p'}} \le c_0 |\Omega|,$$

for all $\beta \le \beta_0$.
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Furthermore, the constant β_0 is sharp.

Here, $1/p + 1/p' = 1$ and

$$\beta_0 = \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma((m+1)/2)}{\Gamma((n-m+1)/2)} \right]^{n/(n-m)}, & m \text{ is odd;} \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(m/2)}{\Gamma((n-m)/2)} \right]^{n/(n-m)}, & m \text{ is even.} \end{cases}$$

Idea of the Proof

Step 1: Riesz potential

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Lemma

Let $m \in 2\mathbb{N}$ *and* $u \in C_c^m(\mathbb{R}^n)$ *then for* $x \in \mathbb{R}^n$

$$u(x) = c_{n,m} \int_{\mathbb{R}^n} \frac{\nabla^m u(y)}{|x-y|^{n-m}} \, dy = c_{n,m} \; I_m * \nabla^m u(x).$$

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Using this lemma it is enough to prove that there exists $C > 0$ such that

$$\sup_{f \in L^p(\Omega), \|f\|_p \le 1} \int_\Omega e^{\beta' |I_m * f|^{p'}} \le C|\Omega|,$$

where $p = n/m$ and $I_m(x) = |x|^{m-n}$.

Step 2: O'Niels lemma

For suitable functions f, g on $\mathbb{R}^n$ there holds

$$(f * g)^*(t) \leq \frac{1}{t} \int_0^t f^*(s)\ ds \int_0^t g^*(s)\ ds + \int_t^\infty f^*(s) g^*(s)\ ds.$$

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Here, f^* denotes the **non-increasing rearrangement** of f defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\},$$

where the distribution function λ_f of f is

$$\lambda_f(s) = |\{|f| > s\}|.$$

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A direct calculation gives us the relation

$$I_m^*(t) \leq \left(\frac{\omega_{n-1}}{nt}\right)^{1/p'},\ for\ t > 0.$$

- Using O'Niel's lemma it follows that

$$\frac{1}{|\Omega|}\int_\Omega e^{\beta|I_m*f(x)|^{p'}}\ dx \leq \frac{1}{|\Omega|}\int_0^{|\Omega|} e^{\beta|I_m*f)^*(t)|^{p'}}\ dt$$

$$\leq \quad \left(\frac{\omega_{n-1}}{nt}\right)^{1/p'}\left(\int_0^t f^*(s)ds + \int_t^{|\Omega|} f^*(s)\ s^{-1/p'}\right).$$

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- Changing the variable as

$$\phi(s) = |\Omega|^{1/p} f^*(|\Omega| e^{-s/p})$$

reduces to the one variable problem of showing the existence of $C_p > 0$ which satisfies

$$\|\phi\|_{L^p(\mathbb{R})} \le 1 \implies \int_0^\infty e^{-F(t)}\ dt \le C_p,$$

where

$$F(t) = t - \int_0^\infty a(s,t)\phi(s)\ ds,$$

$$a(s,t) = p\ e^{(t-s)/p'}, \text{if } s > t.$$

Moser-Trudinger inequality on $\mathbb{R}^n$

- On $\mathbb{R}^n$, Moser-Trudinger inequality does not hold because

$$\int_{\mathbb{R}^n} e^{\alpha|u(x)|^{p'}} \geq \int_{\mathbb{R}^n} 1 = \infty.$$

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- On $\mathbb{R}^n$, Moser-Trudinger inequality does not hold because
$$\int_{\mathbb{R}^n} e^{\alpha|u(x)|^{p'}} \geq \int_{\mathbb{R}^n} 1 = \infty.$$
- One can modify the exponential function and look for inequalities of the form:

There exists $c > 0$ such that
$$\sup_{u\in C_c^k(\mathbb{R}^n), \int_{\mathbb{R}^n}\left(|\nabla^k u|^p+|u|^p\right)\leq 1} \int_{\mathbb{R}^n} \Phi_p(\beta_0|u(x)|^{p'}) < \infty,$$
where
$$\Phi_p(t) = e^t - \sum_{j=0}^{[p]-1} \frac{t^j}{j!}.$$

Question:

What about these inequalities on Riemannian manifolds?

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- G. Lu et al. Sharp Adams inequalities of any fractional order on real hyperbolic spaces of dimension ≥ 3 (Trans. Amer. Math. Soc. 2020).

Our Aim:

To establish sharp Adams type inequalities on the Sobolev spaces $W^{\alpha,n/\alpha}$ of any positive fractional order $\alpha < n$ on Riemannian symmetric spaces of noncompact type of all dimension $n \geq 3$ and of arbitrary rank.

Symmetric spaces of noncompact type:

- Let G be a noncompact, connected, semisimple Lie group with finite center and let K be a maximal compact subgroup of G.

An example

- Let $G = \mathrm{SL}(2,\mathbb{R}), K = \mathrm{SO}(2)$.

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- Let $G = \mathrm{SL}(2, \mathbb{R}), K = \mathrm{SO}(2)$.
- We have $X = G/K \cong \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. The metric on $\mathbb{H}^2$ is given by $ds^2 = \frac{dx^2+dy^2}{y^2}$.

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- We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $A = \exp \mathfrak{a}$. The dimension of $\dim \mathfrak{a}$ is called rank of X.

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- Let Δ denotes the Laplace-Beltrami operator on X.

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- We have $A \equiv \mathbb{R}$. The rank of $\mathbb{H}^2$ is 1.
- $$\Delta f(x, y) = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f.$$

Sobolev space on X

Let $\alpha > 0$ and $1 < p < \infty$. The Sobolev space $W^{\alpha,p}(X)$ is the image of $L^p(X)$ under the operator $(-\Delta)^{-\alpha/2}$, equipped with the norm

$$\|f\|_{W^{\alpha,p}(X)} = \|(-\Delta)^{\alpha/2} f\|_{L^p(X)}.$$

Theorem (M. Bhowmik 2021)

Let $n \geq 3, 0 < \alpha < n$ *and* $p = n/\alpha$*. Suppose* $\Omega \subset X$ *with* $|\Omega| < \infty$*. Then there exists* $C = C(n, \alpha)$ *such that*

$$\frac{1}{|\Omega|} \int_{\Omega} e^{\beta_0 |u(x)|^{p'}} \, dx \leq C,$$

for any $u \in W^{\alpha,p}(X)$ *with* $\int_X |(-\Delta)^{\alpha/2} u(x)|^p \, dx \leq 1$*.*

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Sketch of the proof

- Let $k_{\zeta,\alpha}$ be the kernel of the operator $(-\Delta - |\rho|^2 + \zeta^2)^{-\alpha/2}$, for $0 < \alpha < n$ and $\zeta > 0$.

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For large $|x|$

$$k_{\zeta,\alpha}(x) \lesssim |x|^{(\alpha - l - 1)/2 - |\Sigma_0^+|}\ \phi_0(x)\ e^{-\zeta|x|}.$$

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Lemma

There exists ϵ *satisfying* $0 < \epsilon < \min\{1, n-\alpha\}$ *such that*

$$k_{\zeta,\alpha}(x) \leq \frac{1}{\gamma(\alpha)} \frac{1}{|x|^{n-\alpha}} + \mathcal{O}\left(\frac{1}{|x|^{n-\alpha-\epsilon}}\right),\ \ 0 < |x| < 1,$$

where $\gamma(\alpha) = \frac{2^{\alpha}\ \pi^{n/2}\ \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$, *for* $0 < \alpha < n$.

Lemma

- *For large t*

$$[k_{\zeta,\alpha}]^*(t) \lesssim t^{-1/2-\zeta'/2|\rho|} \ (\log t)^{2|\rho|l/(\zeta'+|\rho|)},$$

where $\zeta' \in (0,\zeta)$.

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- Let $u \in W^{\alpha,p}(X)$ and we write $f = (-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2} u$. Then clearly $u = f * k_{\zeta,\alpha}$ and by the hypothesis $\|f\|_p \leq 1$.

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- We now change the variables

$$\phi(t) = (|\Omega| e^{-t})^{1/p} f^*(|\Omega| e^{-t});$$
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- To get

$$\begin{aligned} & \frac{1}{|\Omega|}\int_\Omega \exp\left(\beta_0(n,\alpha)|u(x)|^{p'}\right)\ dx \\ \le\ & \frac{1}{|\Omega|}\int_0^{|\Omega|} \exp\left(\beta_0(n,\alpha)|u^*(t)|^{p'}\right)\ dt \\ \le\ & \int_0^\infty e^{-F(t)}\ dt, \end{aligned}$$

where $F(t)$ is equals to

$$t - \left(e^t \int_t^\infty e^{-s/p'} \phi(s) ds \int_t^\infty e^{-s/p} \psi(s) ds + \int_{-\infty}^t \phi(s)\psi(s) ds\right)^{p'}.$$

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- By Anker's multiplier theorem

$$\int_X |u(x)|^p \, dx \leq S_p \int_X |(-\Delta - |\rho|^2 + \zeta^2)^{\alpha/2} u(x)|^p \, dx \leq S_p,$$

  provided $\zeta > 2|\rho||1/2 - 1/p|$.
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- On $X \backslash \Omega(u)$

$$\int_{X \backslash \Omega(u)} \Phi_p \left(\beta_0(n,\alpha) |u(x)|^{p'} \right) \, dx \quad \leq \quad \sum_{k=[p]}^{\infty} \frac{\beta_0(n,\alpha)^k}{k!} \|u\|_p^p < \infty.$$

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$$|\Omega(u)| = \int_{\Omega(u)} \; dx \leq \int_X |u(x)|^p \; dx \leq S_p.$$

- On $X \backslash \Omega(u)$

$$\int_{X \backslash \Omega(u)} \Phi_p \left(\beta_0(n,\alpha) |u(x)|^{p'} \right) \; dx \quad \leq \quad \sum_{k=[p]}^{\infty} \frac{\beta_0(n,\alpha)^k}{k!} \|u\|_p^p < \infty.$$

- By first theorem we complete the proof.

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Thank you