HAPPY NEW YEAR
2022

DEDICATION

Prof. Varadharajan Muruganandam

Large time behaviour of heat propagator on Damek–Ricci spaces

Muna Naik

Harish-Chandra Research Institute, Prayagraj
India

5th January, 2022

Joint work with Dr. Rudra P. Sarkar and Dr. Swagato K. Ray

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy = f * m_r(x)$$

where

$$m_r(y) := \frac{1}{|B(o,r)|} \chi_{B(o,r)}(y).$$

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f * m_r(x)$$

where

$$m_r(y) := \frac{1}{|B(o,r)|} \chi_{B(o,r)}(y).$$

- Heat kernel on $\mathbb{R}^n$: $h_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad (t > 0, x \in \mathbb{R}^n).$

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f * m_r(x)$$

where

$$m_r(y) := \frac{1}{|B(o,r)|} \chi_{B(o,r)}(y).$$

- Heat kernel on $\mathbb{R}^n$: $h_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad (t > 0, x \in \mathbb{R}^n)$.
- $u(x,t) = (e^{t\Delta} f)(x) := f * h_t(x)$

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy = f * m_r(x)$$

where

$$m_r(y) := \frac{1}{|B(o,r)|}\chi_{B(o,r)}(y).$$

- Heat kernel on $\mathbb{R}^n$: $h_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad (t > 0, x \in \mathbb{R}^n)$.
- $u(x,t) = (e^{t\Delta} f)(x) := f * h_t(x)$ solves the heat equation:

$$\partial_t u = \Delta u,$$
$$u(x,0) = f(x).$$

Notation:

- Volume mean value operator $B_r f$ is given by

$$B_r f(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f * m_r(x)$$

where

$$m_r(y) := \frac{1}{|B(o,r)|} \chi_{B(o,r)}(y).$$

- Heat kernel on $\mathbb{R}^n$: $h_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad (t > 0, x \in \mathbb{R}^n)$.
- $u(x,t) = (e^{t\Delta} f)(x) := f * h_t(x)$ solves the heat equation:

$$\partial_t u = \Delta u,$$
$$u(x,0) = f(x).$$

Theorem 1 (Repnikov and Éĭdel'man, [5], 1966).

Let $f \in L^\infty(\mathbb{R}^n)$ *and* $x_o \in \mathbb{R}^n$ *be fixed. Then*

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ if and only if } \lim_{t\to\infty} f * h_t(x_o) = L.$$

Theorem 1 (Repnikov and Éĭdel'man, [5], 1966).

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed. Then

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ if and only if } \lim_{t\to\infty} f * h_t(x_o) = L.$$

- The above result was generalized by Li [3] to complete n-dimensional Riemannian manifolds M with nonnegative Ricci curvature satisfying

$$\liminf_{r\to\infty} \frac{|B(x,r)|}{r^n} > 0. \tag{1}$$

Theorem 1 (Repnikov and Éĭdel'man, [5], 1966).

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed. Then

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ if and only if } \lim_{t\to\infty} f * h_t(x_o) = L.$$

- The above result was generalized by Li [3] to complete n-dimensional Riemannian manifolds M with nonnegative Ricci curvature satisfying

$$\liminf_{r\to\infty} \frac{|B(x,r)|}{r^n} > 0. \tag{1}$$

- Using Bishop–Gromov comparison theorem one can show that the geodesic ball $B(x,r)$ has polynomial volume growth.

Theorem 1 (Repnikov and Éĭdel'man, [5], 1966).

Let $f \in L^{\infty}(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed. Then

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ if and only if } \lim_{t\to\infty} f * h_t(x_o) = L.$$

- The above result was generalized by Li [3] to complete n-dimensional Riemannian manifolds M with nonnegative Ricci curvature satisfying

$$\liminf_{r\to\infty} \frac{|B(x,r)|}{r^n} > 0. \tag{1}$$

- Using Bishop–Gromov comparison theorem one can show that the geodesic ball $B(x,r)$ has polynomial volume growth.
- The proof of Li's result relies on the above result of Repnikov et al.

Theorem 2 (Naik–Sarkar–Ray, [4], 2021).

Let S be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_o \in S$,

$$\lim_{r\to\infty} f * m_r(x_o) = L \textit{ implies } \lim_{t\to\infty} f * h_t(x_o) = L,$$

where L is a constant.

Theorem 2 (Naik–Sarkar–Ray, [4], 2021).

Let S be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_o \in S$,

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} f * h_t(x_o) = L,$$

where L is a constant.

- The converse of above theorem is not true in Damek–Ricci space.

Theorem 2 (Naik–Sarkar–Ray, [4], 2021).

Let S be a Damek–Ricci space and $f \in L^\infty(S)$. Then for any $x_o \in S$,

$$\lim_{r\to\infty} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} f * h_t(x_o) = L,$$

where L is a constant.

- The converse of above theorem is not true in Damek–Ricci space.

Question: Can one replace the boundeded ness condition of f in Repnikov and Ĕĭdel'man's Theorem 1 by any other suitable growth condition ?

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g *be measurable functions on* S *such that* $f \in L^\infty(S)$ *and*

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \textit{for almost every } x \in S.$$

Then $\Delta g = 0$.

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g be measurable functions on S such that $f \in L^\infty(S)$ and

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \text{for almost every } x \in S.$$

Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

$$\lim_{s\to\infty} f * h_s(x) = g(x), \tag{2}$$

  for almost every $x \in S$.

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g be measurable functions on S such that $f \in L^\infty(S)$ and

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \textit{for almost every } x \in S.$$

Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

$$\lim_{s\to\infty} f * h_s(x) = g(x), \tag{2}$$

for almost every $x \in S$. Owing to (2) we have

$$\lim_{s\to\infty} f * h_s * h_t(x) = g * h_t(x),$$

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g be measurable functions on S such that $f \in L^\infty(S)$ and

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \textit{for almost every } x \in S.$$

Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

$$\lim_{s\to\infty} f * h_s(x) = g(x), \tag{2}$$

for almost every $x \in S$. Owing to (2) we have

$$\lim_{s\to\infty} f * h_s * h_t(x) = g * h_t(x),$$
$$\Rightarrow \lim_{s\to\infty} f * h_{s+t}(x) = g * h_t(x),$$

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g be measurable functions on S such that $f \in L^\infty(S)$ and

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \textit{for almost every } x \in S.$$

Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

$$\lim_{s\to\infty} f * h_s(x) = g(x), \tag{2}$$

for almost every $x \in S$. Owing to (2) we have

$$\begin{aligned} &\lim_{s\to\infty} f * h_s * h_t(x) = g * h_t(x), \\ &\Rightarrow \lim_{s\to\infty} f * h_{s+t}(x) = g * h_t(x), \\ &\Rightarrow g(x) = g * h_t(x). \end{aligned}$$

Theorem 3 (Naik–Sarkar–Ray, [4], 2021).

Let f, g be measurable functions on S such that $f \in L^\infty(S)$ and

$$\lim_{r\to\infty} f * m_r(x) = g(x), \quad \textit{for almost every } x \in S.$$

Then $\Delta g = 0$.

Proof.

- Applying Theorem 2 we get

$$\lim_{s\to\infty} f * h_s(x) = g(x), \tag{2}$$

for almost every $x \in S$. Owing to (2) we have

$$\begin{aligned} &\lim_{s\to\infty} f * h_s * h_t(x) = g * h_t(x), \\ &\Rightarrow \lim_{s\to\infty} f * h_{s+t}(x) = g * h_t(x), \\ &\Rightarrow g(x) = g * h_t(x). \end{aligned}$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t)$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t)$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t) = \partial_t g$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t) = \partial_t g = 0.$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t) = \partial_t g = 0.$$

□

Proposition 1.

Let $f \in L^\infty(\mathbb{R}^n)$ *and* $x_o \in \mathbb{R}^n$ *be fixed.*

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t) = \partial_t g = 0.$$

□

Proposition 1.

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed. If

$$\lim_{r\to\infty} f * m_r(x_o) = L$$

for a constant L, then for any $x \in \mathbb{R}^n$

$$\lim_{r\to\infty} f * m_r(x) = L.$$

Proof Contd.

Thus $g * h_t = g$ for any $t > 0$.
Hence

$$\Delta g = \Delta(g * h_t) = \partial_t(g * h_t) = \partial_t g = 0.$$

□

Proposition 1.

Let $f \in L^\infty(\mathbb{R}^n)$ and $x_o \in \mathbb{R}^n$ be fixed. If

$$\lim_{r\to\infty} f * m_r(x_o) = L$$

for a constant L, then for any $x \in \mathbb{R}^n$

$$\lim_{r\to\infty} f * m_r(x) = L.$$

Proof

- First assume that $x_o = 0$.

Proof

- First assume that $x_o = 0$.

$$f * m_r(0) - f * m_r(x)$$

Proof

- First assume that $x_o = 0$.

$$
\begin{aligned}
& f * m_r(0) - f * m_r(x) \\
= \quad & \frac{1}{|B(o,r)|} \left(\int_{B(o,r)} f(y)\, dy - \int_{B(x,r)} f(y)\, dy \right)
\end{aligned}
$$

Proof

- First assume that $x_o = 0$.

$$\begin{aligned}
& f * m_r(0) - f * m_r(x) \\
= \; & \frac{1}{|B(o,r)|}\left(\int_{B(o,r)} f(y)\,dy - \int_{B(x,r)} f(y)\,dy\right) \\
= \; & \frac{1}{|B(o,r)|}\left[\left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy\right)\right.
\end{aligned}$$

Proof

- First assume that $x_o = 0$.

$$\begin{aligned} & f * m_r(0) - f * m_r(x) \\ = \; & \frac{1}{|B(o,r)|} \left(\int_{B(o,r)} f(y)\,dy - \int_{B(x,r)} f(y)\,dy \right) \\ = \; & \frac{1}{|B(o,r)|} \left[\left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy \right) \right. \\ & \left. - \left(\int_{B(x,r)\setminus B(o,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy \right) \right] \end{aligned}$$

Proof

- First assume that $x_o = 0$.

$$
\begin{aligned}
& f * m_r(0) - f * m_r(x) \\
= \ & \frac{1}{|B(o,r)|} \left(\int_{B(o,r)} f(y)\,dy - \int_{B(x,r)} f(y)\,dy \right) \\
= \ & \frac{1}{|B(o,r)|} \Bigg[\left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy \right) \\
& \quad - \left(\int_{B(x,r)\setminus B(o,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy \right) \Bigg] \\
= \ & \frac{1}{|B(o,r)|} \left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy - \int_{B(x,r)\setminus B(o,r)} f(y)\,dy \right)
\end{aligned}
$$

Proof

- First assume that $x_o = 0$.

$$
\begin{aligned}
& f * m_r(0) - f * m_r(x) \\
= \; & \frac{1}{|B(o,r)|}\left(\int_{B(o,r)} f(y)\,dy - \int_{B(x,r)} f(y)\,dy\right) \\
= \; & \frac{1}{|B(o,r)|}\left[\left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy\right)\right. \\
& \left. - \left(\int_{B(x,r)\setminus B(o,r)} f(y)\,dy + \int_{B(o,r)\cap B(x,r)} f(y)\,dy\right)\right] \\
= \; & \frac{1}{|B(o,r)|}\left(\int_{B(o,r)\setminus B(x,r)} f(y)\,dy - \int_{B(x,r)\setminus B(o,r)} f(y)\,dy\right)
\end{aligned}
$$

$$|f * m_r(0) - f * m_r(x)|$$

$$|f * m_r(0) - f * m_r(x)| \leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)| \, dy$$

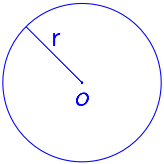
$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r)\triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r)\triangle B(x,r)|}{|B(o,r)|}
\end{aligned}
$$

$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)| \, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.

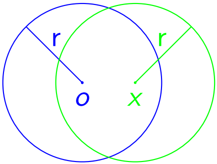
$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r - |x|, r + |x|)$ is the annulus centered at o with inner radius $r - |x|$ and outer radius $r + |x|$.



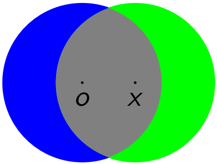
$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r - |x|, r + |x|)$ is the annulus centered at o with inner radius $r - |x|$ and outer radius $r + |x|$.



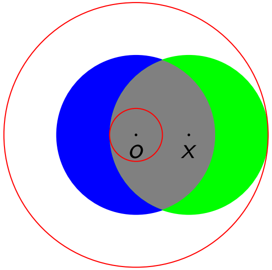
$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.



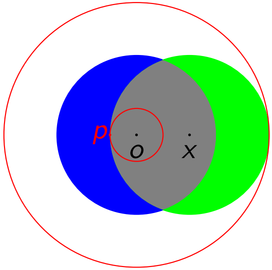
$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.



$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r)\triangle B(x,r)} |f(y)|\,dy \\
&\leq \|f\|_\infty \frac{|B(o,r)\triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

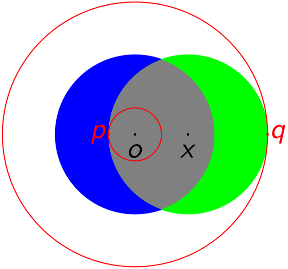
where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.



$op = xp - ox = r - |x|.$

$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r)\triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r)\triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.

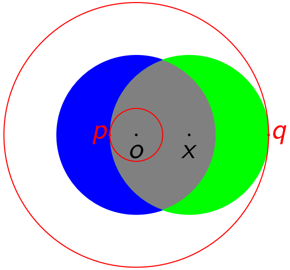


$op = xp - ox = r - |x|.$

$oq = ox + xq = r + |x|.$

$$
\begin{aligned}
|f * m_r(0) - f * m_r(x)| &\leq \frac{1}{|B(o,r)|} \int_{B(o,r) \triangle B(x,r)} |f(y)|\, dy \\
&\leq \|f\|_\infty \frac{|B(o,r) \triangle B(x,r)|}{|B(o,r)|} \\
&\leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}
\end{aligned}
$$

where $A(r-|x|, r+|x|)$ is the annulus centered at o with inner radius $r-|x|$ and outer radius $r+|x|$.



$op = xp - ox = r - |x|.$

$oq = ox + xq = r + |x|.$

$B(o,r) \triangle B(x,r) \subseteq A(r-|x|, r+|x|).$

Thus we get

$$|f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r-|x|, r+|x|)|}{|B(o,r)|}.$$

Thus we get

$$|f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \qquad (3)$$

Since

$$\frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} = \frac{(r + |x|)^n - (r - |x|)^n}{r^n}$$

goes to zero as $r \to \infty$, by taking $\limsup$ in bothsides of (3) we get our desired result. □

Thus we get

$$|f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \tag{3}$$

Since

$$\frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} = \frac{(r + |x|)^n - (r - |x|)^n}{r^n}$$

goes to zero as $r \to \infty$, by taking $\limsup$ in bothsides of (3) we get our desired result. □

- In a Damek–Ricci space,

$$\frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \asymp \frac{e^{\alpha(r+|x|)} - e^{\alpha(r-|x|)}}{e^{\alpha r}} = e^{\alpha|x|} - e^{-\alpha|x|}$$

  doesn't go to zero as $r \to \infty$.

Thus we get

$$|f * m_r(0) - f * m_r(x)| \leq \|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|}. \tag{3}$$

Since

$$\frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} = \frac{(r + |x|)^n - (r - |x|)^n}{r^n}$$

goes to zero as $r \to \infty$, by taking $\limsup$ in bothsides of (3) we get our desired result. □

- In a Damek–Ricci space,

$$\frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \asymp \frac{e^{\alpha(r+|x|)} - e^{\alpha(r-|x|)}}{e^{\alpha r}} = e^{\alpha|x|} - e^{-\alpha|x|}$$

  doesn't go to zero as $r \to \infty$.

Notation:

- We fix the identity element e of the group S as the origin o.

Notation:

- We fix the identity element e of the group S as the origin o.
- Let ρ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.

Notation:

- We fix the identity element e of the group S as the origin o.
- Let ρ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.
- For $\lambda \in \mathbb{C}$, Elementary spherical function φ_λ is the unique smooth radial eigenfunction of Δ with

$$\Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \quad \varphi_\lambda(o) = 1.$$

Notation:

- We fix the identity element e of the group S as the origin o.
- Let ρ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.
- For $\lambda \in \mathbb{C}$, Elementary spherical function φ_λ is the unique smooth radial eigenfunction of Δ with

$$\Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \quad \varphi_\lambda(o) = 1.$$

- $\varphi_\lambda = \varphi_{-\lambda}$ and $\varphi_{i\rho} = \varphi_{-i\rho} = 1$.

Notation:

- We fix the identity element e of the group S as the origin o.
- Let ρ denotes the half of the limit of the mean curvature of geodesic spheres as radius of the sphere tends to infinity.
- For $\lambda \in \mathbb{C}$, Elementary spherical function φ_λ is the unique smooth radial eigenfunction of Δ with

$$\Delta \varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \quad \varphi_\lambda(o) = 1.$$

- $\varphi_\lambda = \varphi_{-\lambda}$ and $\varphi_{i\rho} = \varphi_{-i\rho} = 1$.
- $|\varphi_\lambda(x)| \leq 1$ for $|\Im \lambda| \leq \rho$.

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\,dx.$$

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by
$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\, dx.$$

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\,dx.$$

- For a suitable measure μ on S and $\lambda \in \mathbb{C}$, we define $\widehat{\mu}(\lambda)$ by

$$\widehat{\mu}(\lambda) := \int_S \varphi_\lambda(x)\,d\mu(x).$$

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\, dx.$$

- For a suitable measure μ on S and $\lambda \in \mathbb{C}$, we define $\widehat{\mu}(\lambda)$ by

$$\widehat{\mu}(\lambda) := \int_S \varphi_\lambda(x)\, d\mu(x).$$

- $\widehat{h_t}(\lambda) := e^{-t(\lambda^2+\rho^2)}$.

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\, dx.$$

- For a suitable measure μ on S and $\lambda \in \mathbb{C}$, we define $\widehat{\mu}(\lambda)$ by

$$\widehat{\mu}(\lambda) := \int_S \varphi_\lambda(x)\, d\mu(x).$$

- $\widehat{h_t}(\lambda) := e^{-t(\lambda^2+\rho^2)}$.
- Let $\psi_\lambda(r) := \frac{1}{|B(o,r)|} \int_{B(o,r)} \varphi_\lambda(x)\, dx$

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by
$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\, dx.$$
- For a suitable measure μ on S and $\lambda \in \mathbb{C}$, we define $\widehat{\mu}(\lambda)$ by
$$\widehat{\mu}(\lambda) := \int_S \varphi_\lambda(x)\, d\mu(x).$$
- $\widehat{h_t}(\lambda) := e^{-t(\lambda^2+\rho^2)}$.
- Let $\psi_\lambda(r) := \frac{1}{|B(o,r)|}\int_{B(o,r)} \varphi_\lambda(x)\, dx = \widehat{m_r}(\lambda)$.

$$\left(\text{Recall: } m_r(y) = \frac{1}{|B(o,r)|}\chi_{B(o,r)}(y)\right).$$

Definition

- For $\lambda \in \mathbb{C}$, let $\widehat{f}(\lambda)$ denotes the spherical Fourier transform of f at λ defined by

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)\, dx.$$

- For a suitable measure μ on S and $\lambda \in \mathbb{C}$, we define $\widehat{\mu}(\lambda)$ by

$$\widehat{\mu}(\lambda) := \int_S \varphi_\lambda(x)\, d\mu(x).$$

- $\widehat{h_t}(\lambda) := e^{-t(\lambda^2+\rho^2)}$.
- Let $\psi_\lambda(r) := \frac{1}{|B(o,r)|}\int_{B(o,r)} \varphi_\lambda(x)\, dx = \widehat{m_r}(\lambda)$.

$$\left(\text{Recall: } m_r(y) = \frac{1}{|B(o,r)|}\chi_{B(o,r)}(y)\right).$$

- $\psi_\lambda = \psi_{-\lambda}$ and $\psi_{i\rho} = \psi_{-i\rho} = 1$.

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda)$$

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.
- It is also known that c-function has neither zero nor pole in the region $\Im\lambda < 0$.

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.

- It is also known that c-function has neither zero nor pole in the region $\Im\lambda < 0$.
- Let $0 \neq \alpha \in \mathbb{R}$ be fixed.

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.

- It is also known that c-function has neither zero nor pole in the region $\Im\lambda < 0$.
- Let $0 \neq \alpha \in \mathbb{R}$ be fixed.
- Claim: $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.

- It is also known that c-function has neither zero nor pole in the region $\Im\lambda < 0$.
- Let $0 \neq \alpha \in \mathbb{R}$ be fixed.
- Claim: $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$ and is oscillatory.

- For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of ψ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\psi_\lambda(t) = \mathrm{c}(\lambda) \tag{4}$$

where $\mathrm{c}(\lambda)$ is an analogue of Harish-Chandra c-function.

- It is also known that c-function has neither zero nor pole in the region $\Im\lambda < 0$.
- Let $0 \neq \alpha \in \mathbb{R}$ be fixed.
- Claim: $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$ and is oscillatory.

$$\text{Reason: } e^{i\alpha r} = \frac{1}{e^{-[i(\alpha-i\rho)-\rho]r}\psi_{\alpha-i\rho}(r)}\psi_{\alpha-i\rho}(r).$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

Thus we get $\varphi_\lambda * h_t(x) \quad = \quad \widehat{h_t}(\lambda)\varphi_\lambda(x)$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\text{Thus we get } \varphi_\lambda * h_t(x) = \widehat{h_t}(\lambda)\varphi_\lambda(x) = e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x).$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\begin{aligned} \text{Thus we get } \varphi_\lambda * h_t(x) &= \widehat{h_t}(\lambda)\varphi_\lambda(x) \\ &= e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x). \\ \text{Hence } \varphi_{\alpha-i\rho} * h_t(x) &= e^{-t[(\alpha-i\rho)^2+\rho^2]}\varphi_{\alpha-i\rho}(x) \end{aligned}$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\begin{aligned} \text{Thus we get } \varphi_\lambda * h_t(x) &= \widehat{h_t}(\lambda)\varphi_\lambda(x) \\ &= e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x). \\ \text{Hence } \varphi_{\alpha-i\rho} * h_t(x) &= e^{-t[(\alpha-i\rho)^2+\rho^2]}\varphi_{\alpha-i\rho}(x) \\ &= e^{-t(\alpha^2-\rho^2-2i\alpha\rho+\rho^2)}\varphi_{\alpha-i\rho}(x) \end{aligned}$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\begin{aligned}
\text{Thus we get } \varphi_\lambda * h_t(x) &= \widehat{h_t}(\lambda)\varphi_\lambda(x) \\
&= e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x). \\
\text{Hence } \varphi_{\alpha-i\rho} * h_t(x) &= e^{-t[(\alpha-i\rho)^2+\rho^2]}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-\rho^2-2i\alpha\rho+\rho^2)}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-2i\alpha\rho)}\varphi_{\alpha-i\rho}(x)
\end{aligned}$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\begin{aligned}
\text{Thus we get } \varphi_\lambda * h_t(x) &= \widehat{h_t}(\lambda)\varphi_\lambda(x) \\
&= e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x). \\
\text{Hence } \varphi_{\alpha-i\rho} * h_t(x) &= e^{-t[(\alpha-i\rho)^2+\rho^2]}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-\rho^2-2i\alpha\rho+\rho^2)}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-2i\alpha\rho)}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t\alpha^2}e^{2it\alpha\rho}\varphi_{\alpha-i\rho}(x).
\end{aligned}$$

- For a radial measure μ, one can show that

$$\varphi_\lambda * \mu(x) = \widehat{\mu}(\lambda)\varphi_\lambda(x).$$

$$\begin{aligned}
\text{Thus we get } \varphi_\lambda * h_t(x) &= \widehat{h_t}(\lambda)\varphi_\lambda(x) \\
&= e^{-t(\lambda^2+\rho^2)}\varphi_\lambda(x). \\
\text{Hence } \varphi_{\alpha-i\rho} * h_t(x) &= e^{-t[(\alpha-i\rho)^2+\rho^2]}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-\rho^2-2i\alpha\rho+\rho^2)}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t(\alpha^2-2i\alpha\rho)}\varphi_{\alpha-i\rho}(x) \\
&= e^{-t\alpha^2}e^{2it\alpha\rho}\varphi_{\alpha-i\rho}(x).
\end{aligned}$$

- From above, it is clear that for any $x \in S$, $\varphi_{\alpha-i\rho} * h_t(x) \to 0$ as $t \to \infty$.

Similarly we have, $\varphi_\lambda * m_r(x) \quad = \quad \widehat{m_r}(\lambda)\varphi_\lambda(x)$

Similarly we have,
$$\begin{aligned}\varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x)\\ &= \psi_\lambda(r)\varphi_\lambda(x).\end{aligned}$$

$$\begin{aligned}
\text{Similarly we have, } \varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x) \\
&= \psi_\lambda(r)\varphi_\lambda(x). \\
\text{Thus we get, } \varphi_{\alpha - i\rho} * m_r(x) &= \psi_{\alpha - i\rho}(r)\varphi_{\alpha - i\rho}(x).
\end{aligned}$$

$$\begin{aligned} \text{Similarly we have, } \varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x) \\ &= \psi_\lambda(r)\varphi_\lambda(x). \\ \text{Thus we get, } \varphi_{\alpha-i\rho} * m_r(x) &= \psi_{\alpha-i\rho}(r)\varphi_{\alpha-i\rho}(x). \end{aligned}$$

Since $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$,

$$\begin{aligned} \text{Similarly we have, } \varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x) \\ &= \psi_\lambda(r)\varphi_\lambda(x). \\ \text{Thus we get, } \varphi_{\alpha-i\rho} * m_r(x) &= \psi_{\alpha-i\rho}(r)\varphi_{\alpha-i\rho}(x). \end{aligned}$$

Since $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$, it follows that $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value as $r \to \infty$.

$$\begin{aligned} \text{Similarly we have, } \varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x) \\ &= \psi_\lambda(r)\varphi_\lambda(x). \\ \text{Thus we get, } \varphi_{\alpha-i\rho} * m_r(x) &= \psi_{\alpha-i\rho}(r)\varphi_{\alpha-i\rho}(x). \end{aligned}$$

Since $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$, it follows that $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value as $r \to \infty$.
But since $\varphi_{\alpha-i\rho} * h_t(x) \to 0$ as $t \to \infty$,

$$\begin{aligned} \text{Similarly we have, } \varphi_\lambda * m_r(x) &= \widehat{m_r}(\lambda)\varphi_\lambda(x) \\ &= \psi_\lambda(r)\varphi_\lambda(x). \\ \text{Thus we get, } \varphi_{\alpha-i\rho} * m_r(x) &= \psi_{\alpha-i\rho}(r)\varphi_{\alpha-i\rho}(x). \end{aligned}$$

Since $\psi_{\alpha-i\rho}(r)$ does not converge to any value as $r \to \infty$, it follows that $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value as $r \to \infty$.
But since $\varphi_{\alpha-i\rho} * h_t(x) \to 0$ as $t \to \infty$, it follows that the function $\varphi_{\alpha-i\rho}$ forms a counterexample for Repnikov et al's theorem in Damek–Ricci space.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\, v(x)$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\,v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\, v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\, v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\,v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\,v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$. Clearly f is a strictly positive function

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\,v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$. Clearly f is a strictly positive function and $f * m_r(x)$ does not converge to any value as $r \to \infty$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\, v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$. Clearly f is a strictly positive function and $f * m_r(x)$ does not converge to any value as $r \to \infty$.
- On the other hand, since

$$u * h_t(x) + i\, v * h_t(x) = \varphi_{\alpha-i\rho} * h_t(x) \to 0$$

as $t \to \infty$,

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\, v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$. Clearly f is a strictly positive function and $f * m_r(x)$ does not converge to any value as $r \to \infty$.
- On the other hand, since

$$u * h_t(x) + i\, v * h_t(x) = \varphi_{\alpha-i\rho} * h_t(x) \to 0$$

as $t \to \infty$, it is clear that $u * h_t(x) \to 0$ as $t \to \infty$.

- Let $\varphi_{\alpha-i\rho}(x) = u(x) + i\,v(x)$. As $\varphi_{\alpha-i\rho} * m_r(x)$ does not converge to any value, it follows that either $u * m_r(x)$ or $v * m_r(x)$ does not converge to any value as $r \to \infty$. Without loss of generality assume that $u * m_r(x)$ does not converge.
- Since $|\varphi_{\alpha-i\rho}(x)| \leq 1$, $|u(x)| \leq 1$.
- Let $f(x) = 2 - u(x)$. Clearly f is a strictly positive function and $f * m_r(x)$ does not converge to any value as $r \to \infty$.
- On the other hand, since

$$u * h_t(x) + i\,v * h_t(x) = \varphi_{\alpha-i\rho} * h_t(x) \to 0$$

as $t \to \infty$, it is clear that $u * h_t(x) \to 0$ as $t \to \infty$. Consequently we get

$$\lim_{t\to\infty} f * h_t(x) = 2 - \lim_{t\to\infty} u * h_t(x) = 2.$$

for any fixed $x \in S$.

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \text{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$.

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \text{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_o \in S$,

$$\lim_{r \to \infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t \to \infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \textit{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \textit{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \textit{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_o \in S$,

$$\lim_{r \to \infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \textit{ implies } \lim_{t \to \infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is not true for all complex number λ with nonzero real and imaginary parts,

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \text{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is not true for all complex number λ with nonzero real and imaginary parts, i.e. $\lambda \notin (i\mathbb{R} \cup \mathbb{R})$.

Theorem 4 (Naik–Sarkar–Ray, [4], 2021).

Let f be a measurable function on S satisfying

$$|f(x)| \leq Ae^{B|x|}, \quad \text{for almost every } x \in S, \tag{5}$$

and for constants $A > 0$ and $B \in \mathbb{R}$. Then for any $\lambda \in i\mathbb{R}$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is not true for all complex number λ with nonzero real and imaginary parts, i.e. $\lambda \notin (i\mathbb{R} \cup \mathbb{R})$.

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$.

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t\left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \textit{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is also not true for all complex number λ with nonzero real and imaginary parts,

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is also not true for all complex number λ with nonzero real and imaginary parts, i.e. $\lambda \notin (i\mathbb{R} \cup \mathbb{R})$.

- For $p \in [1, \infty)$, we define the weak L^p space by

$$L^{p,\infty}(X) := \{f : S \to \mathbb{C} \text{ measurable} \mid \sup_{t>0} t \left|\{x \mid |f(x)| > t\}\right|^{1/p} < \infty\}.$$

Theorem 5 (Naik–Sarkar–Ray, [4], 2021).

Fix a $p > 2$. Let $\lambda = \pm i(2/p - 1)\rho$. Then for $f \in L^{p,\infty}(S)$ and a point $x_o \in S$,

$$\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x_o) = L \text{ implies } \lim_{t\to\infty} e^{t(\lambda^2+\rho^2)} f * h_t(x_o) = L,$$

where L is a constant.

- Converse of the above theorem is not true.
- The above theorem is also not true for all complex number λ with nonzero real and imaginary parts, i.e. $\lambda \notin (i\mathbb{R} \cup \mathbb{R})$.

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$.

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$. *Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$. *Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

(a) $|f(x)| \leq B\varphi_\lambda(x)$ *for almost every* $x \in S$, *for a constant* $B > 0$,

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$*. Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

(a) $|f(x)| \leq B\varphi_\lambda(x)$ *for almost every* $x \in S$*, for a constant* $B > 0$*,*

(b) $f \in L^{p,\infty}(S)$ *for some* $p \in (2, \infty)$ *satisfying* $|\lambda| = (1 - 2/p)\rho$*,*

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$. *Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

(a) $|f(x)| \leq B\varphi_\lambda(x)$ *for almost every* $x \in S$, *for a constant* $B > 0$,

(b) $f \in L^{p,\infty}(S)$ *for some* $p \in (2, \infty)$ *satisfying* $|\lambda| = (1 - 2/p)\rho$,

(c) $f \in L^\infty(S)$ *and* $|\lambda| = \rho$.

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$. *Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

(a) $|f(x)| \leq B\varphi_\lambda(x)$ *for almost every* $x \in S$, *for a constant* $B > 0$,

(b) $f \in L^{p,\infty}(S)$ *for some* $p \in (2, \infty)$ *satisfying* $|\lambda| = (1 - 2/p)\rho$,

(c) $f \in L^\infty(S)$ *and* $|\lambda| = \rho$.

If $\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x) = g(x)$, *for almost every* $x \in S$, *then*

$$\Delta g = -(\lambda^2 + \rho^2)g.$$

Theorem 6 (Naik–Sarkar–Ray, [4], 2021).

Fix a $\lambda \in i\mathbb{R}$. *Let* f, g *be measurable functions on* S *such that* f *satisfies one of these three conditions:*

(a) $|f(x)| \leq B\varphi_\lambda(x)$ *for almost every* $x \in S$, *for a constant* $B > 0$,

(b) $f \in L^{p,\infty}(S)$ *for some* $p \in (2, \infty)$ *satisfying* $|\lambda| = (1 - 2/p)\rho$,

(c) $f \in L^\infty(S)$ *and* $|\lambda| = \rho$.

If $\lim_{r\to\infty} \frac{1}{\psi_\lambda(r)} f * m_r(x) = g(x)$, *for almost every* $x \in S$, *then*

$$\Delta g = -(\lambda^2 + \rho^2)g.$$

Estimate of heat kernel

Estimate of heat kernel

$$h_t(r) \asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n = \dim(S), r \geq 0)$$

Estimate of heat kernel

$$
\begin{aligned}
h_t(r) &\asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n = \dim(S), r \geq 0) \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{\rho r} e^{-2\rho r}
\end{aligned}
$$

Estimate of heat kernel

$$
\begin{aligned}
h_t(r) &\asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n=\dim(S), r \geq 0) \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{\rho r} e^{-2\rho r} \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(r-2\rho t)^2}{4t}} e^{-2\rho r}.
\end{aligned}
$$

Estimate of heat kernel

$$
\begin{aligned}
h_t(r) &\asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n=\dim(S), r \geq 0) \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{\rho r} e^{-2\rho r} \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(r-2\rho t)^2}{4t}} e^{-2\rho r}.
\end{aligned}
$$

- Exponential factors:

Estimate of heat kernel

$$
\begin{aligned}
h_t(r) &\asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n = \dim(S), r \geq 0) \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{\rho r} e^{-2\rho r} \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(r-2\rho t)^2}{4t}} e^{-2\rho r}.
\end{aligned}
$$

- Exponential factors:

$$\mathbb{R}^n : \quad p_t(r) = e^{-\frac{r^2}{4t}}, \quad \text{(peak is always at 0).}$$

Estimate of heat kernel

$$
\begin{aligned}
h_t(r) &\asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{-\rho r}, \quad (n=\dim(S), r \geq 0) \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{r^2}{4t}} e^{-\rho^2 t} e^{\rho r} e^{-2\rho r} \\
&= t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(r-2\rho t)^2}{4t}} e^{-2\rho r}.
\end{aligned}
$$

- Exponential factors:

$$\mathbb{R}^n: \quad p_t(r) = e^{-\frac{r^2}{4t}}, \ \text{(peak is always at 0).}$$

$$S: \quad p_t(r) = e^{-\frac{(r-2\rho t)^2}{4t}}, \ \text{(peak is at } 2\rho t\text{).}$$

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$.

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n : \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n: \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

$$S: \quad \lim_{t\to\infty} \int_{|r-2\rho t|>s(t)} h_t(r)\, dr = 0.$$

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n : \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

$$S : \quad \lim_{t\to\infty} \int_{|r-2\rho t|>s(t)} h_t(r)\, dr = 0.$$

- This means that heat produced initially at the origin $o \in S$ does not diffuse homogeneously but concentrates asymptotically in an annulus of width $s(t)$ moving to infinity with speed 2ρ.

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n : \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

$$S : \quad \lim_{t\to\infty} \int_{|r-2\rho t|>s(t)} h_t(r)\, dr = 0.$$

- This means that heat produced initially at the origin $o \in S$ does not diffuse homogeneously but concentrates asymptotically in an annulus of width $s(t)$ moving to infinity with speed 2ρ.
- Such behavior sharply contrasts to that of the Euclidean space $\mathbb{R}^n$.

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n : \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

$$S : \quad \lim_{t\to\infty} \int_{|r-2\rho t|>s(t)} h_t(r)\, dr = 0.$$

- This means that heat produced initially at the origin $o \in S$ does not diffuse homogeneously but concentrates asymptotically in an annulus of width $s(t)$ moving to infinity with speed 2ρ.
- Such behavior sharply contrasts to that of the Euclidean space $\mathbb{R}^n$.
- The above result was first proved by Davies et al for hyperbolic spaces [2]

Let $s(t)$ be a positive function such that $\frac{s(t)}{\sqrt{t}} \to \infty$ as $t \to \infty$. Then

$$\mathbb{R}^n : \quad \lim_{t\to\infty} \int_{r>s(t)} h_t(r)\, dr = 0.$$

$$S : \quad \lim_{t\to\infty} \int_{|r-2\rho t|>s(t)} h_t(r)\, dr = 0.$$

- This means that heat produced initially at the origin $o \in S$ does not diffuse homogeneously but concentrates asymptotically in an annulus of width $s(t)$ moving to infinity with speed 2ρ.
- Such behavior sharply contrasts to that of the Euclidean space $\mathbb{R}^n$.
- The above result was first proved by Davies et al for hyperbolic spaces [2] and by Anker and Setti for all symmetric spaces of noncompact type [1].

Reference

[1] Anker, J. P.; Setti, A. G. *Asymptotic finite Propagation speed for heat diffusion on Certain Riemannian Manifolds*. Journal of Functional Analysis, 103, 50–61 (1992).

[2] Davies, E. B.;*Heat kernels and spectral theory*. Cambridge University Press, Cambridge, 1989.

[3] Li, P.; *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*. Ann. of Math. (2) 124 (1986), no. 1, 1–21.

[4] Naik, M., Sarkar R., Ray, S. K. *Large time behavior of heat propagator*. Bull. Sci. math. 167 (2021) 102955.

[5] Repnikov, V. D.; Èĭdel'man, S. D. *Necessary and sufficient conditions for establishing a solution to the Cauchy problem*. Soviet Math. Dokl. 7 (1966), 388–391.

Thank You