An uncertainty principle on the Heisenberg group

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Based on a joint work with Prof. S. Thangavelu

- **What is an Uncertainty Principle?**
  - In harmonic analysis, uncertainty principles says that a function f and its Fourier transform $\hat{f}$ defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx$$

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  can not both be SMALL simultaneously.
  - Depending on various notions of smallness, we can get different uncertainty principles.
  - **Example:** (Hardy) Let $f(x) = O(e^{-a|x|^2})$ and $\hat{f}(\xi) = O(e^{-b|\xi|^2})$. If $ab > 1/4$ then $f = 0$.

An uncertainty principle

- Let $f \in L^1(\mathbb{R}^n)$ be such that $|\hat{f}(\xi)| \le e^{-a|\xi|}$. Then the inversion formula for the Fourier transform yields:

$$f(x+iy) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(x+iy).\xi} d\xi.$$

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  Equivalently one can also ask:
- Question: For a non-trivial function f which is compactly supported, what would be the best possible decay admissible for $\hat{f}$?
- In 1934, Ingham gave an answer to this question for the one dimensional case (i.e., on $\mathbb{R}$)

Ingham's theorem

Theorem

Let $\theta : [0, \infty) \to [0, \infty)$ *be decreasing function vanishing at infinity. There exists a non-trivial* $f \in C_c(\mathbb{R})$ *satisfying*

$$|\hat{f}(\xi)| \leq Ce^{-\theta(|\xi|)|\xi|}$$

if and only if $\int_1^\infty \theta(t)t^{-1}dt < \infty$.

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- When $\int_1^\infty \theta(t)t^{-1}dt = \infty$, the function $t\theta(t)$ is unbounded on $(1,\infty)$. So, 'Ingham type condition' imposes a certain amount of rapid decay on Fourier transform.

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- The above theorem says that when $\int_1^\infty \theta(t)t^{-1}dt = \infty$, the Fourier transform of a compactly supported continuous function cannot have Ingham type decay!

What Ingham did...

Ingham type decay

Quasi analyticity of the function

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Denjoy-Carleman theorem

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$C\{M_n\} := \{f \in C^\infty(\mathbb{R}) : \|f^{(n)}\|_\infty \leq C_f A^n M_n\}$ is quasi-analytic iff $\sum_{n=1}^\infty M_n^{-1/n} = \infty$.

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Let $\theta : [0, \infty) \to [0, \infty)$ be decreasing function vanishing at infinity. Let $f \in L^1(\mathbb{R}^n)$ be such that its Fourier transform satisfies $|\hat{f}(\xi)| \leq Ce^{-\theta(|\xi|)|\xi|}$.

1. *If $\int_1^\infty \theta(t)t^{-1}dt = \infty$ and f vanishes on an open set, then $f = 0$.*
2. *If $\int_1^\infty \theta(t)t^{-1}dt < \infty$, then there exists $f \in C_c^\infty(\mathbb{R}^n)$ whose Fourier transform satisfies the above estimate.*

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- In 1978, P.R. Chernoff came up with a nice sufficient condition for smooth functions on $\mathbb{R}^n$ to be quasi-analytic.

Chernoff's theorem

Theorem (P.R. Chernoff, 1978)

Let f be a smooth function on $\mathbb{R}^n$. Let $\Delta_{\mathbb{R}^n}$ be the standard Laplacian on $\mathbb{R}^n$. Assume that $\Delta_{\mathbb{R}^n}^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|\Delta_{\mathbb{R}^n}^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at $x_0 \in \mathbb{R}^n$, then f is identically zero.

- It is worth noting that proving Ingham's theorem requires essentially a version of the previous theorem with a stronger vanishing condition that the function f vanishes on an open set.

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- M.Bhowmik, S.Pusti, S.K.Ray, Theorems of Ingham and Chernoff on Riemannian symmetric spaces of noncompact type, JFA, 2020.

- Consider the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t).(w, s) := \left(z + w, t + s + \frac{1}{2}\operatorname{Im}(z.\bar{w})\right).$$

This is a step two Nilpotent Lie group where the Lebesgue measure *dzdt* plays the role of Haar measure.

Harmonic analysis on the Heisenberg group

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This is a step two Nilpotent Lie group where the Lebesgue measure *dzdt* plays the role of Haar measure.

- The group Fourier transform of $f \in L^1(\mathbb{H}^n)$ is an operator valued function defined on non-zero reals by

$$\widehat{f}(\lambda) := \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)dzdt$$

where π_λ's are Schrödinger representaion of $\mathbb{H}^n$ given by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

- **Plancherel formula:** It is known that when $f \in L^1 \cap L^2(\mathbb{H}^n)$ its Fourier transform is actually a Hilbert-Schmidt operator and one has

$$\int_{\mathbb{H}^n} |f(z,t)|^2 dz dt = (2\pi)^{-(n+1)} \int_{-\infty}^{\infty} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda.$$

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- The Heisenberg Lie algebra, $\mathfrak{h}_n$ consists of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n+1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \;\; Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \;\; j = 1, 2, ..., n, \text{ and } T = \frac{\partial}{\partial t}.$$

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- The **full Laplacian** on $\mathbb{H}^n$ is defined by

$$\Delta_{\mathbb{H}} := -\sum_{j=1}^{\infty} (X_j^2 + Y_j^2) - T^2$$

- An analogue of the situation $\widehat{\Delta_{\mathbb{R}^n} f}(\xi) = |\xi|^2 \widehat{f}(\xi)$, in the context of $\mathbb{H}^n$ is given by

$$\widehat{\Delta_{\mathbb{H}} f}(\lambda) = (H(\lambda) + \lambda^2) \widehat{f}(\lambda)$$

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where $H(\lambda) := -\Delta_{\mathbb{R}^n} + \lambda^2 |x|^2$. As a mater of fact, an analogue of Ingham type decay condition on $\mathbb{R}^n$ viz. $|\hat{f}(\xi)| \leq e^{-\theta(|\xi|)|\xi|}$ $(i.e., \overline{\hat{f}(\xi)}\hat{f}(\xi) \leq e^{-2\theta(|\xi|)|\xi|})$, in the setting of $\mathbb{H}^n$ takes the form

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C\, e^{-2|\lambda|\,\Theta(|\lambda|)} e^{-2\sqrt{H(\lambda)}\,\Theta(\sqrt{H(\lambda)})}, \;\; \lambda \neq 0 \qquad (1)$$

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We have the following exact analogue of Ingham's theorem in the setting of $\mathbb{H}^n$.

Theorem (Ganguly-Thangavelu)

Let $\Theta(\lambda)$ be a nonnegative function on $[0, \infty)$ such that $\Theta(\lambda)$ decreases to zero when $\lambda \to \infty$. Then there exists a nonzero compactly supported continuous function f on $\mathbb{H}^n$ whose Fourier transform $\widehat{f}(\lambda)$ satisfies the estimate (1) if and only if $\int_1^\infty \Theta(t) t^{-1} dt < \infty$.

Two main results of this talk

Theorem (Ganguly-Thangavelu)

Let $\Theta(\lambda)$ be a nonnegative function on $[0,\infty)$ such that it decreases to zero when $\lambda \to \infty$ and satisfies the conditions $\int_1^\infty \Theta(t) t^{-1} dt = \infty$. Let f be an integrable function on $\mathbb{H}^n$ whose Fourier transform satisfies the estimate

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C\, e^{-2|\lambda|\,\Theta(|\lambda|)} e^{-2\sqrt{H(\lambda)}\,\Theta(\sqrt{H(\lambda)})}, \;\; \lambda \neq 0.$$

Then f cannot vanish on any nonempty open set unless it is identically zero.

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In order to prove this, we need to use an analogue of Chernoff's theorem for the full Laplacian $\Delta_{\mathbb{H}}$:

Theorem (Ganguly-Thangavelu)

Let $f \in C^\infty(\mathbb{H}^n)$ be such that $\Delta_{\mathbb{H}}^m f \in L^2(\mathbb{H}^n)$ for all $m \geq 0$ and satisfies the Carleman condition $\sum_{m=1}^\infty \|\Delta_{\mathbb{H}}^m f\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a non-empty open set, then f is identically zero.

Time for proofs!

Chernoff's theorem for the full Laplacian

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Sketch of proof:

WLOG, we can assume that f vanishes on an open set $B_a(0) \times (-a, a) \subset V$ containing the origin.

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•Reduction to the radial case:

- Fix $z \in V$ and consider the spherical mean

$$F_z(r,t) := f * \mu_r(z,t) := \int_{|w|=r} f\left(z - w, t - \frac{1}{2}\operatorname{Im} z \cdot \overline{w}\right) d\mu_r(w)$$

where μ_r is the normalised surface measure on the sphere $\{(z,t) \in \mathbb{H}^n : |z| = r\}$.

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- Choose $\delta = \min(a/2, \sqrt{a})$. Then, $z \in B_\delta(0)$, $F_z(r,t) = 0$ for all $(r,t) \in (0,\delta) \times (-\delta/2, \delta/2) =: U \subset S := \mathbb{R}_+ \times \mathbb{R}$.

- Considering F_z as a radial function on $\mathbb{H}^n$, using the relation $\Delta_{\mathbb{H}} f * \mu_r(z,t) = \Delta_S F_z(r,t)$ we can show that

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |\Delta_S^m F_z(r,t)|^2 r^{2n-1} drdt \leq C_n \int_{\mathbb{H}^n} |\Delta_{\mathbb{H}}^m f(z,t)|^2 dzdt$$

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where

$$\Delta_S := -\frac{\partial^2}{\partial r^2} - \frac{2n-1}{r}\frac{\partial}{\partial r} - \frac{1}{4} r^2 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial t^2}.$$

- As a matter of fact, $\sum_{m=1}^{\infty} \|\Delta_{\mathbb{H}}^m f\|_2^{-\frac{1}{2m}} = \infty$ implies $\sum_{m=1}^{\infty} \|\Delta_S^m F_z\|_2^{-\frac{1}{2m}} = \infty$. Moreover, we have

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Theorem

Let $g \in C^\infty(S)$ be such that $\Delta_S^m f \in L^2(S)$ for all $m \geq 0$. Assume that $\sum_{m=1}^{\infty} \|\Delta_S^m g\|_2^{-\frac{1}{2m}} = \infty$. If f vanishes on a neighbourhood of the origin, then f is identically zero.

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- $f * \mu_r(z,t) = F_z(r,t) = 0$ for all $z \in B_\delta(0)$ and $t \in \mathbb{R}$.
- As a result,

$$0 = \int_{-\infty}^{\infty} f * \mu_r(z,t) e^{i\lambda t} dt = f^\lambda *_\lambda \mu_r(z)$$

where

$$f^\lambda *_\lambda \mu_r(z) = \int_{|w|=r} f^\lambda(z-w) e^{\frac{i\lambda}{2} Im(z.\bar{w})} d\sigma(w).$$

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Moreover,

$$Q_k^\lambda f^\lambda *_\lambda \mu_r(z) = c_k^\lambda(r) Q_k^\lambda f^\lambda$$

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- **Chernoff's theorem for Δ_S**

Theorem (de Jeu, Ann. of Prob.)

Let μ be a finite positive Borel measure on $\mathbb{R}^n$ for which all the moments $M^{(j)}(m) = \int_{\mathbb{R}^n} x_j^m d\mu(x), m \geq 0$ are finite. If we further assume that the moments satisfy the Carleman condition $\sum_{m=1}^{\infty} M^{(j)}(2m)^{-1/2m} = \infty$, $j = 1, 2, ..., n$, then polynomials are dense in $L^p(\mathbb{R}^n, d\mu), 1 \leq p < \infty$.

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Under the assumption that

$$\sum_{m=1}^{\infty} \|\Delta_S^m g\|_2^{-\frac{1}{2m}} = \infty, \; g(r,t) = 0, \; (r,t) \in U$$

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- We define a measure μ_g on $\mathbb{R}^2$ supported on the Heisenberg fan

$$\Omega := \{(\lambda, (2k+n)|\lambda|) : \lambda \in \mathbb{R}, k \in \mathbb{N}\}$$

in such a way that

$$\int_{\mathbb{R}^2} \varphi(x,-y) d\mu_g(x,y) = \int_{\mathbb{R}^2} \varphi(x,y) d\mu_g(x,y).$$

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- Carleman condition for Δ_S would imply the Carleman condition for the even moments.—the hypothesis of de Jeu's theorem is satisfied!
- Finally, using the density of polynomials in $L^1(\mathbb{R}^2)$ and $g = 0$ on U, we can show that $g = 0$.

Towards Ingham's theorem on the Heisenberg group

Theorem (Bagchi-Ganguly-Sarkar-Thangavelu)

Assume that Θ is a positive decreasing function on $[0, \infty)$, vanishing at infinity for which $\int_1^\infty \Theta(t)\, t^{-1} dt < \infty$. Then we can construct a compactly supported nontrivial smooth radial function f on $\mathbb{H}^n$ whose Fourier transform satisfies the estimate

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq C e^{-2\sqrt{H(\lambda)}\,\Theta(\sqrt{H(\lambda)})}$$

where $H(\lambda) = -\Delta + \lambda^2 |x|^2$ is the scaled Hermite operator.

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1. Let $f_j := \rho_j^{-2n} \chi_{B(0, A\rho_j)}$ and $g_j := (2\tau_j^2)^{-1} \chi_{(-\tau_j^2, \tau_j^2)}$. Suppose $F_j(z,t) := f_j(z) g_j(t)$.

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1. Let $f_j := \rho_j^{-2n}\chi_{B(0,A\rho_j)}$ and $g_j := (2\tau_j^2)^{-1}\chi_{(-\tau_j^2,\tau_j^2)}$. Suppose $F_j(z,t) := f_j(z)g_j(t)$. $f := F_1 * F_2 * F_3 * \ldots$.
2. f is Heisenberg radial and hence $\widehat{f}(\lambda) = \sum_{k=0}^\infty R_k^\lambda(f)P_k(\lambda)$. Also

$$|R_k^\lambda(f)|^2 \leq Ce^{-2\sqrt{(2k+n)|\lambda|}}\Theta(\sqrt{(2k+n)|\lambda|}).$$

Let $g \in C_c(\mathbb{R})$ whose Euclidean Fourier transform satisfies the estimate

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where $\Theta_\delta(\lambda) = \delta\Theta(\lambda\delta)$ which inherits the properties of Θ. Moreover, $\{F_\delta\}_{\delta>0}$ forms an approximate identity!

Theorem

Let $\Theta(\lambda)$ be a nonnegative function on $[0,\infty)$ such that it decreases to zero when $\lambda \to \infty$ and satisfies the conditions $\int_1^\infty \Theta(t)t^{-1}dt = \infty$. Let f be an integrable function on $\mathbb{H}^n$ whose Fourier transform satisfies the estimate

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Then f cannot vanish on any nonempty open set unless it is identically zero.

Sketch of proof:

- Using the Plancherel formula we have

$$\|\Delta_{\mathbb{H}}^m f\|_2^2 = \int_{-\infty}^{\infty} \|\widehat{f}(\lambda)(H(\lambda)+\lambda^2)^m\|_{HS}^2 |\lambda|^n d\lambda.$$

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- Under the assumption that $\Theta(\lambda) \geq c\lambda^{-1/2}$ we obtain the estimate

$$\|\Delta_{\mathbb{H}}^m f\|_2 \leq C^{2m}\Big(\frac{m}{\Theta(m^4)}\Big)^{2m}$$

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$$\widehat{f * g_\delta}(\lambda)^* \widehat{f * g_\delta}(\lambda) \leq Ce^{-2\sqrt{H(\lambda)}\,\Psi_\delta(\sqrt{(H(\lambda)})}\; e^{-2|\lambda|\,\Psi_\delta(|\lambda|)}$$

  where $\Psi_\delta(\lambda) = \Theta(\lambda) + \theta_\delta(\lambda)$ for which the extra condition viz. $\Psi_\delta(\lambda) \geq c_\delta|\lambda|^{-1/2}, |\lambda| \geq 1$ holds.

- So, the Fourier transform of the function $f * g_\delta$ satisfies

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- **Open question:**
  - Is it possible to prove a weaker analogue of Chernoff's theorem for the sublaplacian?
  - Is it possible to prove an exact analogue of Chernoff's theorem for the full Laplacian or the sublaplacian?

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THANK YOU!