A classification of isometries of infinite dimensional hyperbolic space

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17th Discussion Meeting in Harmonic Analysis

NISER, Bhubneshwar

January 6, 2022

Conformally invariant metric

Poincar$\acute{e}$ metric

Let Δ denote the open unit ball in $\mathbb{C}$ and ρ, the Poincar$\acute{e}$ metric. For $z, w \in \Delta$,

$$\rho(z,w) = \frac{1}{2} log \frac{1 + \left|\dfrac{w-z}{1-\overline{z}w}\right|}{1 - \left|\dfrac{w-z}{1-\overline{z}w}\right|}$$

Harnack inequality

If $u : \overline{B}(a; R) \longrightarrow \mathbb{R}$ is continuous, harmonic in $B(a; R)$, and $u \geq 0$ then for $0 \leq r < R$ and all θ

$$\frac{R-r}{R+r}U(a) \leq u(a + re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$$

[1] H. S. Bear and W. Smith, A tale of two conformally invariant metrics, J. Math. Anal. Appl. **318** (2006), no. 2, 498–506. MR2215165.

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Harnack metric [1]

For $z, w \in \overline{B}(a;\, R)$

$$d(z,w) = sup\Big\{|log\, u(z) - log\, u(w)| \;\Big|\; \text{u is positive and harmonic in } \overline{B}(a;\, R)\Big\}.$$

Holomorphic self maps on Δ are distance decreasing for Poincar$\acute{e}$ metric and this property forms the statement of Schwarz-Pick Lemma.

[1] H. S. Bear and W. Smith, A tale of two conformally invariant metrics, J. Math. Anal. Appl. **318** (2006), no. 2, 498–506. MR2215165.

Hyperbolic geometry of the n dimesnional spaces

Let B^n denote the open unit ball in $\mathbb{C}^n$.

Carathéodory metric [2]

Let Hol(B^n, Δ) denote the set of all holomorphic mappings $f : B^n \to \Delta$. Then

$$C_{B^n}(x, y) = \sup_f \rho(f(x), f(y)) \quad \text{for all } x,\, y \in B^n.$$

[2]Kobayashi, S. *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften, 318, Springer-Verlag, Berlin, 1998.

z-classes [1] - Two elements in a group are said to be z-equivalent if their centralizers are conjugate.

[1] R. S. Kulkarni, Dynamical types and conjugacy classes of centralizers in groups, J. Ramanujan Math. Soc. **22** (2007), no. 1, 35–56. MR2312547

[2] Gongopadhyay, Krishnendu; Kulkarni, Ravi S. z-classes of isometries of the hyperbolic space. Conform. Geom. Dyn. 13 (2009), 91–109. MR2491719

[3] Cirici, J. Classification of isometries of spaces of constant curvature and invariant subspaces, Linear Algebra Appl. **450** (2014), 250–279. MR3192481

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1. Gongopadhyay and Kulkarni [2]
2. Joana Cirici [3].

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$Aut(B^n) \cong U(1,n)/Z(U(1,n))$

Fixed point classification of $Aut(B^n)$

An element of $Aut(B^n)$ is called

- Elliptic if it has one fixed point in B^n.
- Hyperbolic if not elliptic and has exactly 2 fixed points on ∂B^n.
- Parabolic if not elliptic and has a unique fixed point on ∂B^n.

z-classes of $Aut(B^n)$ (Chen and Greenberg[1], Gongopadhyay and Kulkarni [4])

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5. Elliptic and hyperbolic isometries decompose the space into corresponding eigenspaces. Hence their centralizers will preserve each eigenspace and vice versa.
6. Parabolic isometries are not semisimple. A parabolic isometry T is conjugate to pe. p is strictly parabolic and e is unitary. Also an isometry S commutes with T if and only if it commutes with both p and e. Minimal polynomial and characterstic polynomial determine the conjugacy of a parabolic isometry.

Generalization to infinite dimensional setup

Franzoni and Vesentini [4] have discussed the hyperbolic structure for infinite dimensional setup.

Holomorphicity in infinite dimensions

Let E and F be two complex normed spaces. Let U be an open subset of E. A mapping $f : U \longrightarrow F$ is called an F-valued holomorphic function if for every $a \in U$, there exists $A \in L(E, F)$ such that
$$\lim_{x \to a} \frac{f(x) - f(a) - A(x-a)}{\|x-a\|} = 0.$$

[4] Franzoni, Tullio; Vesentini, Edoardo. Holomorphic maps and invariant distances. Notas de Matemática [Mathematical Notes], 69. North-Holland Publishing Co., Amsterdam-New York, 1980. viii+226 pp. ISBN: 0-444-85436-3 MR0563329

H - Infinite dimensional Hilbert space.
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Holomorphic self maps on B are distance decreasing for the Carath$\acute{e}$odory metric. So holomorphic bijections become isometries.

Group of isometries

$AutB$ - Group of holomorphic bijections on B.

General element of $AutB$ - $U \circ f_{x_0}$, $x_0 \in B$,

U is a unitary operator on H and f_{x_0} is some holomorphic bijection on B.

$f_{x_0} : B_{x_0} \longrightarrow H$ defined by

$$f_{x_0}(x) = T_{x_0}\left(\frac{x - x_0}{1 - \langle x, x_0 \rangle}\right),$$

$B_{x_0} = \left\{ x \in H \ : \ \|x\| < \frac{1}{\|x_0\|} \right\}$. $f_{x_0} \restriction_B$ is a bi-holomorphic surjection.

$T_{x_0} : H \longrightarrow H$ is a linear map expressed as

$$T_{x_0}(x) = \frac{\langle x, x_0 \rangle}{1 + \sqrt{1 - \|x_0\|^2}} x_0 + \sqrt{1 - \|x_0\|^2}\, x.$$

$Aut(B)$ is known to act transitively on B.

Linear representation of $AutB$

$\mathcal{A}$ - sesquilinear form on $H \oplus \mathbb{C}$.

$$\mathcal{A}((x,\lambda),\,(y,\mu)) = \langle x, y\rangle - \lambda\overline{\mu}.$$

G - Group of all bijective linear operators on $H \oplus \mathbb{C}$ prserving $\mathcal{A}$.

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G - Group of all bijective linear operators on $H \oplus \mathbb{C}$ prserving $\mathcal{A}$.

For $T \in G$, T has the form $\begin{bmatrix} A & \xi \\ \left\langle \cdot, \dfrac{A^*(\xi)}{a} \right\rangle & a \end{bmatrix}$ where $\xi \in H$ satisfies

$$A^*A = I + \frac{1}{|a|^2}\left\langle \cdot, A^*(\xi)\right\rangle A^*(\xi)$$

and

$$|a|^2 = 1 + \|\xi\|^2.$$

The center Z_G of G - $\{e^{i\theta}I, \;\; \theta \in \mathbb{R}\}$.

Theorem (Franzoni and Vesentini [2])

The map $\phi : G \to$ Aut(B) defined by $\phi(T) = \tilde{T}$ is an onto homomorphism where $T = \begin{bmatrix} A & \xi \\ \left\langle \cdot, \dfrac{A^*(\xi)}{a} \right\rangle & a \end{bmatrix}$ and $\tilde{T} = \dfrac{A(\cdot) + \xi}{\left\langle \cdot, \frac{A^*(\xi)}{a} \right\rangle + a}$.

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Hence $\tilde{\phi} : G/Z_G \to Aut(B)$ is an onto isomorphism.

Fixed point classification of isometries

Theorem (Hayden and Suffridge [9])

If $g \in \mathrm{Aut}(\mathrm{B})$ has no fixed point in B, then the fixed point set in $\overline{\mathrm{B}}$ consists of one or two points.

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Let Q be the corresponding quadratic form of the sesquilinear form $\mathcal{A}$.

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We call a vector $x \in H \oplus \mathbb{C}$ *time like* if $Q(x) < 0$, *light like* if $Q(x) = 0$ and *space like* if $Q(x) > 0$.

Observation - Observe that for $x \in \overline{B}$, x is a fixed point for an isometry in $AutB$ if and only if $(x, 1)$ is an eigenvector for the corresponding element in G.

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Simplified form of elements of G

Proposition

A general element of G is of the form $e^{i\theta}\begin{bmatrix} UA & U(\xi) \\ \langle\cdot,\xi\rangle & a \end{bmatrix}$, $\theta \in \mathbb{R}$, where $\xi \in H$, $a = \sqrt{1+\|\xi\|^2}$, U is a unitary operator on H and A is a positive operator on H such that $A = I$ on $\langle\xi\rangle^{\perp}$ and $A(\xi) = a\xi$.

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For $T=\begin{bmatrix} UA & U(\xi) \\ \langle\cdot,\xi\rangle & a \end{bmatrix}$,

$T^{-1}=\begin{bmatrix} (UA)^* & -\xi \\ -\langle\cdot,U(\xi)\rangle & a \end{bmatrix}$ and $T^*=\begin{bmatrix} (UA)^* & \xi \\ \langle\cdot,U(\xi)\rangle & a \end{bmatrix}$

Examples of isometries based on fixed point classification

Subclass of G having a two-dimensional reducing subspace

Let $T = \begin{bmatrix} UA & U(\xi) \\ \langle \cdot, \xi \rangle & a \end{bmatrix} \in G$. If U preserves $\langle \xi \rangle$, then T preserves $\langle \xi \rangle \oplus \mathbb{C}$. Hence $T = T_1 \oplus T_2$ where $T_1 = T \upharpoonright_{\langle \xi \rangle \oplus \mathbb{C}}$ and $T_2 = U \upharpoonright_{\langle \xi \rangle^{\perp}}$.

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Spectrum

Let $T = \begin{bmatrix} UA & r\xi \\ \langle \cdot, \xi \rangle & a \end{bmatrix} \in G$, $|r| = 1$. Then $sp(T) = \{\lambda_1, \lambda_2\} \cup sp(U \restriction_{\langle \xi \rangle^{\perp}})$ where $\lambda_1, \lambda_2 = \dfrac{a(r+1) \pm \sqrt{a^2(r+1)^2 - 4r}}{2}$ respectively. The eigenspaces corresponding to the eigenvalues λ_1 and λ_2 are generated by the eigenvectors $(k_1\xi, 1)$ and $(k_2\xi, 1)$ where $k_1, k_2 = \dfrac{a(r-1) \pm \sqrt{a^2(r+1)^2 - 4r}}{2\|\xi\|^2}$ respectively.

Proposition (M. M. Mishra and A.)

Let $T = \begin{bmatrix} UA & r\xi \\ \langle \cdot, \xi \rangle & a \end{bmatrix} \in G$, $|r| = 1$ be such that $T = T_1 \oplus T_2$. If the eigenvalues of T_1 are distinct, then T is elliptic for $r = -1$ and hyperbolic for $r \neq -1$. For identical eigenvalues, T is parabolic.

Normal isometries

Proposition (M. M. Mishra and A.)

1. A unitary element of G is of the form $e^{i\theta}\begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}$, where V is a unitary operator on H and $\theta \in \mathbb{R}$.
2. A normal element of G is of the form $e^{i\theta}\begin{bmatrix} UA & \xi \\ \langle \cdot, \xi \rangle & a \end{bmatrix}$, $\theta \in \mathbb{R}$.
3. For S normal and defined as above in point (2), $\sigma(S) = \{a \pm \|\xi\|\} \cup \sigma\left(U\big|_{\langle \xi \rangle^{\perp}}\right)$, where $a \pm \|\xi\|$ are both positive, non unit modulus and inverses of each other. Eigenspaces corresponding to the eigenvalues $a \pm \|\xi\|$ are spanned by the eigenvectors $\left(\pm\frac{\xi}{\|\xi\|}, 1\right)$ respectively.
4. Normal isometries are hyperbolic in nature.

Conjugacy results

Elliptic isometry (M. M. Mishra and A.)

Let T be an elliptic isometry in G.

1. Then T has a time like eigenvector and viceversa.
2. $T = T_1 \oplus T_2$ with respect to the sesquilinear form $\mathcal{A}$ where $T_1 = T\big|_{\langle (x,1) \rangle}$, (x,1) is time like eigenvector and $T_2 = T\big|_{\langle (x,1) \rangle^{\perp}}$, $\mathcal{A}\big|_{\langle (x,1) \rangle^{\perp}}$ is unitary.
3. T is conjugate to a unitary operator.

Hyperbolic isometry (M. M. Mishra and A.)

Let T be a hyperbolic element in G.

1. $T = T_1 \oplus T_2$ with respect to the sesquilinear form $\mathcal{A}$ where $T_1 = T\big|_{\langle (x,1),(y,1)\rangle}$, $(x, 1)$ and $(y, 1)$ are light like eigenvectors and $T_2 = T\big|_{\langle (x,1),(y,1)\rangle^{\perp}}$, $\mathcal{A}\big|_{\langle (x,1),(y,1)\rangle^{\perp}}$ is unitary.
2. T is conjugate to a normal isometry.
3. Spectrum of T has only two non-unit modulus values, which are eigenvalues, positive and inverses of each other. Eigenspaces with respect to these eigenvalues are one dimensional spaces, each generated by a light like eigenvector. Rest of the spectral values lie on S^1.

Subclass of Parabolic isometry

Heisenberg translations

A Heisenberg translation $T \in G$ decomposes $H \oplus \mathbb{C}$ into a finite dimensional subspace K and its orthogonal complement. Dimension of K is either 2 or 3. $\sigma(T) = \sigma_{pt}(T) = \{\lambda\}$, $|\lambda| = 1$. Degree of the minimial polynomial of $T \restriction_K$ is either 2 or 3. $T \restriction_{K^\perp} = \lambda I$.

Spectral theory of normal operators [5]

Let H be a separable Hilbert space.

1. Result 1- (Unitary equivalence of normal operators) If N is a normal operator on H, there are mutually singular measures μ_∞, μ_1, μ_2... and an isomorphism $U : H \longrightarrow L^2(\mu_\infty; H_\infty) \oplus L^2(\mu_1) \oplus L^2(\mu_2; H_2) \oplus ...$ such that $UNU^{-1} = N_\infty \oplus N_1 \oplus N_2 \oplus ...$ where H_n is an n-dimensional Hilbert space, $L^2(\mu_n; H_n)$ is the space of square integrable H_n valued functions and N_n is multiplication by z on $L^2(\mu_n; H_n)$.

[5] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1985. MR0768926

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2. Result 2- (Centralizer of a normal operator) Also, if N is multiplication by z on $L^2(\mu; H_n)$, then

$$\{N\}' = \{M_\phi \ : \ \phi \in L^\infty(\mu; B(H_n))\}.$$

This gives

$$\{N_\infty \oplus N_1 \oplus N_2 \oplus \ldots\}' = L^\infty(\mu_\infty; B(H_\infty)) \oplus L^\infty(\mu_1) \oplus L^\infty(\mu_2; B(H_2)) \oplus \ldots$$

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Centralizers

Lemma (Centralizer of a unitary operator)

If V is a unitary operator on a separable Hilbert space H, then

$$Z(V) = U^{-1} Z(V_\infty \oplus V_1 \oplus V_2 \oplus ...)U$$

where U is as in the preceding theorem and $Z(V_\infty \oplus V_1 \oplus V_2 \oplus ...) = U(H_\infty) \oplus U(H_1) \oplus U(H_2) \oplus ...$, $U(H_n)$ is the group of unitary elements in $L^\infty(\mu; B(H_n))$.

For a subspace $K \subseteq H$, let $G(\mathcal{A} \restriction_K)$ denote the collection of bijective linear isometries with respect to the sesquilinear form $\mathcal{A} \restriction_K$.

Elliptic isometry (M. M. Mishra and A.)

Let T be an elliptic isometry decomposing the infinite dimensional space $H \oplus \mathbb{C}$ into a time like finite dimensional sub-eigenspace say K, and $K^\perp$ with respect to the sesquilinear form $\mathcal{A}$. Then $Z(T) = Z(T \restriction_K) \times Z(T \restriction_{K^\perp})$ where $Z(T \restriction_K) = G(\mathcal{A} \restriction_K)$.

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Hyperbolic isometry (M. M. Mishra and A.)

Let T be a hyperbolic isometry decomposing the space $\mathcal{H} \oplus \mathbb{C}$ into a two-dimensional subspace say L, generated by two light like eigenvectors say x and y, and $L^\perp$ with respect to the sesquilinear form $\mathcal{A}$. Then $Z(T) = Z(T \restriction_L) \times Z(T \restriction_{L^\perp})$ where $Z(T \restriction_L) = S^1 \cup \mathbb{R}^+$.

Hyperbolic geometry for unit ball in $B(K, H)$

Bear [6] defined the notion of Gleason part to an arbitrary linear subspace B of $C(X)$.
For x, $y \in X$, $x \sim y\ (a)$ if and only if

$$\frac{1}{a} < \frac{u(x)}{u(y)} < a$$

for all strictly positive functions $u \in B$.
$x \sim y$ if $x \sim y\ (a)$ for some a.
The metric D on each part is defined as follows.
For x and y in the same part,

$$D(x, y) = log\, R(x, y).$$

where $R(x, y) = inf\{a \ :\ x \sim y\ (a)\}$.

[6] H. S. Bear, A geometric characterization of Gleason parts, Proc. Amer. Math. Soc. **16** (1965), 407–412. MR0181910

Ion Suciu [7], For two contractions T_1 and T_2, $T_1 \sim T_2$ if there exists $a \in (0,1)$ such that

$$a \operatorname{Re} p(T_1) \leq \operatorname{Re} p(T_2) \leq a^{-1} \operatorname{Re} p(T_1)$$

for each complex valued polynomial p with positive real part.

[7] I. Suciu, Analytic relations between functional models for contractions, Acta Sci. Math. (Szeged) **34** (1973), 359–365. MR0320783

[8] C. Foiaș, On Harnack parts of contractions, Rev. Roumaine Math. Pures Appl. **19** (1974), 315–318. MR0348537

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Group of isometries

Franzoni, Tullio. The group of holomorphic automorphisms in certain J^*-algebras. Ann. Mat. Pura Appl. (4) 127 (1981), 51–66. MR0633394

B_1- Open unit ball in $B(K, H)$, K and H are Hilbert spaces.

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B_1- Open unit ball in $B(K, H)$, K and H are Hilbert spaces.

For $h \in AutB_1$,

$$h = T_B \circ L,$$

$T_B \in Aut(B_1)$,

$$T_B(A) = (I - BB^*)^{-\frac{1}{2}}(A + B)(I + B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}$$

and L is a surjective linear isometry defined on $B(K, H)$ as

$$L(A) = UAV,\ \ V \in U(K),\ U \in U(H).$$

Linearization of $Aut(B_1)$

$U(K,H)$- Group of all bijective linear transformations defined on $H \oplus K$ preserving the following hermitian form. $\mathcal{M} : (H \oplus K) \times (H \oplus K) \to \mathbb{C}$ as

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For $T \in U(K,H)$, $T = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ satisfying
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Franzoni [3]

$\psi : U(K,H) \to Aut(B_1)$ is an onto homomorphism defined as $T \mapsto \psi(T)$ where $\psi(T)(A) = (BA + C)(DA + E)^{-1}$.

Simplified form of elements of $U(K,H)$

Let $K = \mathbb{C}^n$.

Proposition (M. M. Mishra and A.)

For, $T \in U(K,H)$, $T = \begin{bmatrix} BU & CV \\ DU & EV \end{bmatrix}$, $U \in U(H)$, $V \in U(K)$, B and E are positive invertible in $B(H)$ and $B(K)$ respectively.
$E(e_i) = a_i\, e_i$, $a_i > 0$,
$C(e_i) = \xi_i$,
$D = C^* = 0$ on $(Ran\, C)^\perp$ and $C^*(\xi_i) = \|\xi_i\|^2 e_i$.
${a_i}^2 = 1 + \|\xi_i\|^2$ and $\langle \xi_i, \xi_j \rangle = 0$, $i \neq j$, $i, j \in \{1, 2, ..., n\}$.
$B = I$ on $(Ran\, C)^\perp$ and $B(\xi_i) = a_i\, \xi_i$

Let $\xi_i \neq 0$, $i \in \{1, 2, ..., k\}$ and $\xi_j = 0$, $j \in \{k+1, ..., n\}$.

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Let $C_i \oplus E_i = H_i$.
$H \oplus K = H_1 \oplus H_2 \oplus ... \oplus H_l \oplus (Ran\, C)^{\perp} \oplus K_2$.

Normal isometry

Proposition (M. M. Mishra and A.)

$T = \begin{bmatrix} BU & CV \\ DU & EV \end{bmatrix}$ is a normal isometry if and only if

1. U preserves each C_i.
2. V preserves each E_i.
3. $[U \restriction_{C_i}] = [V \restriction_{E_i}]$.

So, For a normal isometry T, $T = T_1 \oplus T_2 \oplus \ldots \oplus T_l \oplus T' \oplus T''$ where $T_i = T \restriction_{H_i}$, $T' = U \restriction_{(Ran\, C)^{\perp}}$ and $T'' = V \restriction_{K^{\perp}}$.

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Spectrum

For T normal, $\sigma(T) = \cup\sigma(T_i) \cup \sigma(U \restriction_{(Ran\, C)^\perp}) \cup \sigma(V \restriction_{K^\perp})$
$\sigma(T_i) = \{\lambda_i\mu_1, \lambda_2\mu_2, ..., \lambda_{p_i}\mu_{p_i}\}$ where $\{\mu_1, \mu_2, .., \mu_{p_i}\} = \sigma(U \restriction_{C_i})$ and $\{\lambda_1, \lambda_2, ..., \lambda_{p_i}\} = \{a_i \pm \|\xi\|\}$.

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Thank you.