On sharp weighted L^p-estimates for pseudo-multipliers associated to Grushin operators

Riju Basak
(Based on a joint work with Sayan Bagchi, Rahul Garg and Abhishek Ghosh)

17th DMHA, NISER Bhubaneswar
January 5-8, 2022

- In this talk, I shall discuss some analogues of Mihlin-Hörmander multiplier type theorems for pseudo-multipliers associated with Grushin operators.

- In this talk, I shall discuss some analogues of Mihlin-Hörmander multiplier type theorems for pseudo-multipliers associated with Grushin operators.
- In order to motivate our work, let me first do a short survey on related results in the Euclidean space.

- In this talk, I shall discuss some analogues of Mihlin-Hörmander multiplier type theorems for pseudo-multipliers associated with Grushin operators.
- In order to motivate our work, let me first do a short survey on related results in the Euclidean space.
- The Fourier multiplier operator T_m associated to $m \in L^\infty(\mathbb{R}^n)$ is defined by

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi,$$

for suitable functions f on $\mathbb{R}^n$, where $\widehat{f}$ stands for the Fourier transform of f.

- In this talk, I shall discuss some analogues of Mihlin-Hörmander multiplier type theorems for pseudo-multipliers associated with Grushin operators.

- In order to motivate our work, let me first do a short survey on related results in the Euclidean space.

- The Fourier multiplier operator T_m associated to $m \in L^\infty(\mathbb{R}^n)$ is defined by

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi,$$

  for suitable functions f on $\mathbb{R}^n$, where $\widehat{f}$ stands for the Fourier transform of f.

- By the use of the Plancherel Theorem it is easy to see that T_m is bounded on $L^2(\mathbb{R}^n)$.

Mihlin-Hörmander multiplier theorem

- For $p \neq 2$, we need some regularity on m for T_m to be bounded on $L^p(\mathbb{R}^n)$.

Mihlin-Hörmander multiplier theorem

- For $p \neq 2$, we need some regularity on m for T_m to be bounded on $L^p(\mathbb{R}^n)$.

Theorem (Mihlin-Hörmander multiplier theorem)

Let m be a smooth function such that

$$\left|\partial_\xi^\alpha m(\xi)\right| \lesssim_\alpha (1+|\xi|)^{-|\alpha|}$$

for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq \lfloor n/2 \rfloor + 1$. Then the operator T_m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Pseudo-differential operators on the Euclidean space

- The pseudo-differential operator $m(x, D)$ associated to $m \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$m(x, D)f(x) := \int_{\mathbb{R}^n} m(x, \xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi,$$

for suitable functions f on $\mathbb{R}^n$.

Pseudo-differential operators on the Euclidean space

- The pseudo-differential operator $m(x, D)$ associated to $m \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$m(x, D)f(x) := \int_{\mathbb{R}^n} m(x, \xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi,$$

  for suitable functions f on $\mathbb{R}^n$.

- $m \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is not sufficient to guarantee the boundedness of the operator $m(x, D)$ on $L^2(\mathbb{R}^n)$.

Definition (Symbol class $S^{\sigma}_{\rho,\delta}$)

For any $\sigma \in \mathbb{R}$ and $\rho, \delta \geq 0$, we define

$$S^{\sigma}_{\rho,\delta} := \Big\{ m \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n) : \left| \partial_x^{\beta} \partial_{\xi}^{\alpha} m(x,\xi) \right| \lesssim_{\alpha,\beta} (1 + |\xi|)^{\sigma - \rho|\alpha| + \delta|\beta|}, \quad \forall\, \alpha, \beta \in \mathbb{N}^n \Big\}.$$

Definition (Symbol class $S^{\sigma}_{\rho,\delta}$)

For any $\sigma \in \mathbb{R}$ and $\rho, \delta \geq 0$, we define

$$S^{\sigma}_{\rho,\delta} := \Big\{ m \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n) : \left| \partial_x^{\beta} \partial_{\xi}^{\alpha} m(x,\xi) \right| \lesssim_{\alpha,\beta} (1 + |\xi|)^{\sigma - \rho|\alpha| + \delta|\beta|}, \quad \forall\, \alpha, \beta \in \mathbb{N}^n \Big\}.$$

Theorem (Calderón-Vaillancourt, 1971)

Let $m \in S^0_{\rho,\delta}$, with $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then the operator $m(x, D)$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a bounded operator from $L^2(\mathbb{R}^n)$ to itself.

Weighted Boundedness on Euclidean space

Definition (A_p class)

- Let $1 < p < \infty$. A weight w is said to be in class A_p if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \; dx \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} dx \right)^{p-1} < \infty,$$

  where the supremum is over all cubes with sides parallel to coordinates axes.

- A weight w is said to be in class A_1 if

$$[w]_{A_1} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \; dx \right) \|w^{-1}\|_{L^\infty(Q)} < \infty,$$

  where the supremum is over all cubes with sides parallel to coordinates axes.

- In 1982, Miller studied the boundedeness of the pseudo-differential operator $m(x,D)$ on $L^p(w)$.

Theorem (N. Miller, 1982)

Let m be such that

$$\left|\partial_x^\beta \partial_\xi^\alpha m(x,\xi)\right| \lesssim_{\alpha,\beta} (1+|\xi|)^{-|\alpha|},$$

for all $\alpha,\beta \in \mathbb{N}^n$ *such that* $|\alpha| \leq n+1$, $|\beta| \leq 1$. *Also assume that* $m(x,D)$ *is bounded on* $L^2(\mathbb{R}^n)$. *Then for* $1 < p < \infty$, $m(x,D)$ *has a bounded extension to* $L^p(w)$ *for* $w \in A_p$.

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- There are L^p boundedness results for pseudo-multipliers for the set up beyond Euclidean spaces.

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- There are L^p boundedness results for pseudo-multipliers for the set up beyond Euclidean spaces.

  ▶ Heisenberg group : H. Bahouri-C. Fermanian-Kammerer-I. Gallagher (2012).

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- There are L^p boundedness results for pseudo-multipliers for the set up beyond Euclidean spaces.

  ▶ Heisenberg group : H. Bahouri-C. Fermanian-Kammerer-I. Gallagher (2012).

  ▶ Graded Lie group: V. Fischer-M. Ruzhansky, "Quantization on nilpotent Lie groups".

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- There are L^p boundedness results for pseudo-multipliers for the set up beyond Euclidean spaces.

  - Heisenberg group : H. Bahouri-C. Fermanian-Kammerer-I. Gallagher (2012).

  - Graded Lie group: V. Fischer-M. Ruzhansky, "Quantization on nilpotent Lie groups".

  - Compact Lie group : M. Ruzhansky - V. Turunen, "Pseudo-differential operators and symmetries".

- Recently, D. Beltran and L. Cladek have proved sparse domination results for pseudo-differential operators on R^n.

- There are L^p boundedness results for pseudo-multipliers for the set up beyond Euclidean spaces.

  ▶ Heisenberg group : H. Bahouri-C. Fermanian-Kammerer-I. Gallagher (2012).

  ▶ Graded Lie group: V. Fischer-M. Ruzhansky, "Quantization on nilpotent Lie groups".

  ▶ Compact Lie group : M. Ruzhansky - V. Turunen, "Pseudo-differential operators and symmetries".

  ▶ Homogeneous space associated with a class of self adjoint operator : F. Bernicot and D. Frey (2014).

Hermite operator

- The Hermite operator H on $\mathbb{R}^n$ is given by

$$H = -\Delta + |x|^2$$

where Δ stands for the standard Laplacian on R^n.

Hermite operator

- The Hermite operator H on $\mathbb{R}^n$ is given by

$$H = -\Delta + |x|^2$$

where Δ stands for the standard Laplacian on R^n.

- The spectral resolution of H is given by

$$H = \sum_{k=0}^{\infty} (2k+n) P_k$$

where P_k stands for the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the eigenspace for the Hermite operator corresponding to the eigenvalue $2k+n$.

Pseudo-multipliers associated to Hermite operator

- Given $m \in L^\infty(\mathbb{R}^n \times \mathbb{N})$, one can (densely) define the Hermite pseudo-multiplier on $L^2(\mathbb{R}^n)$ by

$$m(x, H) := \sum_{k=0}^{\infty} m(x, 2k+n) P_k.$$

Pseudo-multipliers associated to Hermite operator

- Given $m \in L^\infty(\mathbb{R}^n \times \mathbb{N})$, one can (densely) define the Hermite pseudo-multiplier on $L^2(\mathbb{R}^n)$ by

$$m(x, H) := \sum_{k=0}^{\infty} m(x, 2k+n) P_k.$$

- In 1996, Epperson first studied the Hermite pseudo-multipliers. He proved L^P-boundedness results for the operator $m(x, H)$ in dimension $n = 1$.

Pseudo-multipliers associated to Hermite operator

- Given $m \in L^{\infty}(\mathbb{R}^n \times \mathbb{N})$, one can (densely) define the Hermite pseudo-multiplier on $L^2(\mathbb{R}^n)$ by

$$m(x, H) := \sum_{k=0}^{\infty} m(x, 2k+n) P_k.$$

- In 1996, Epperson first studied the Hermite pseudo-multipliers. He proved L^P-boundedness results for the operator $m(x, H)$ in dimension $n = 1$.

- In 2015, Bagchi and Thangavelu studied the same problem in higher dimensions. They actually prove weighted boundednes results for these operators.

Results for Hermite pseudo-multipliers

- Let $\Delta_d m(x,k) = m(x,k+1) - m(x,k)$, and $\Delta_d^l m = \Delta_d(\Delta_d^{l-1} m)$, $l \geq 2$.

Results for Hermite pseudo-multipliers

- Let $\Delta_d m(x,k) = m(x,k+1) - m(x,k)$, and $\Delta_d^l m = \Delta_d(\Delta_d^{l-1} m)$, $l \geq 2$.

Theorem (Bagchi-Thangavelu, 2015)

Assume that the Hermite pseudo-multiplier $m(x,H)$ is bounded on $L^2(\mathbb{R}^n)$. Suppose $\sup_{x\in\mathbb{R}^n} \left|\Delta_d^l m(x,k)\right| \leq C_l(2k+n)^{-l}$ for $l = 1,2,\ldots,\lfloor n/2 \rfloor + 1$. And also assume that partial derivatives $\frac{\partial}{\partial x_j} m(x,k)$ satisfy same estimates for $l = 1,2,\ldots,\lfloor n/2 \rfloor$. Then for any $2 < p < \infty$ and $w \in A_{p/2}$ we have

$$\int_{\mathbb{R}^n} |m(x,H)f(x)|^p \, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, w(x)\, dx$$

for all $f \in L^p(\mathbb{R}^n, wdx)$.

Results for Hermite pseudo-multipliers

- Let $\Delta_d m(x,k) = m(x,k+1) - m(x,k)$, and $\Delta_d^l m = \Delta_d(\Delta_d^{l-1} m)$, $l \geq 2$.

Theorem (Bagchi-Thangavelu, 2015)

Assume that the Hermite pseudo-multiplier $m(x,H)$ is bounded on $L^2(\mathbb{R}^n)$. Suppose $\sup_{x\in\mathbb{R}^n} \left|\Delta_d^l m(x,k)\right| \leq C_l(2k+n)^{-l}$ for $l = 1,2,\ldots,\lfloor n/2\rfloor + 1$. And also assume that partial derivatives $\frac{\partial}{\partial x_j} m(x,k)$ satisfy same estimates for $l = 1,2,\ldots,\lfloor n/2\rfloor$. Then for any $2 < p < \infty$ and $w \in A_{p/2}$ we have

$$\int_{\mathbb{R}^n} |m(x,H)f(x)|^p \, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \, w(x)\, dx$$

for all $f \in L^p(\mathbb{R}^n, wdx)$.

- By increasing the values of l upto $n+1$, they also prove that $m(x,H)$ is bounded on $L^p(\mathbb{R}^n, wdx)$ for $1 < p < \infty$, $w \in A_p$.

Grushin operators

- For any $\varkappa \in \mathbb{N}_+$, the Grushin operator $G_\varkappa$ on $\mathbb{R}^{n_1+n_2}$ is given by

$$G_\varkappa = -\Delta_{x'} - |x'|^{2\varkappa}\Delta_{x''},$$

where $\Delta_{x'}$, $\Delta_{x''}$ are standard Laplacians on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively.

Grushin operators

- For any $\varkappa \in \mathbb{N}_+$, the Grushin operator $G_\varkappa$ on $\mathbb{R}^{n_1+n_2}$ is given by

$$G_\varkappa = -\Delta_{x'} - |x'|^{2\varkappa}\Delta_{x''},$$

  where $\Delta_{x'}$, $\Delta_{x''}$ are standard Laplacians on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively.

- Consider the following first order gradient vector fields:

$$X_j = \frac{\partial}{\partial x'_j} \quad \text{and} \quad X_{\alpha,k} = x'^{\alpha}\frac{\partial}{\partial x''_k}, \quad \text{for } 1 \leq j \leq n_1, 1 \leq k \leq n_2$$

  and $\alpha \in \mathbb{N}^{n_1}$ with $|\alpha| = \varkappa$.

Grushin operators

- For any $\varkappa \in \mathbb{N}_+$, the Grushin operator $G_\varkappa$ on $\mathbb{R}^{n_1+n_2}$ is given by

$$G_\varkappa = -\Delta_{x'} - |x'|^{2\varkappa}\Delta_{x''},$$

where $\Delta_{x'}$, $\Delta_{x''}$ are standard Laplacians on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively.

- Consider the following first order gradient vector fields:

$$X_j = \frac{\partial}{\partial x'_j} \quad \text{and} \quad X_{\alpha,k} = x'^{\alpha}\frac{\partial}{\partial x''_k}, \quad \text{for } 1 \leq j \leq n_1, 1 \leq k \leq n_2$$

and $\alpha \in \mathbb{N}^{n_1}$ with $|\alpha| = \varkappa$.

- We denote the gradient vector field $(X_j, X_{\alpha,k})_{1\leq j\leq n_1,\, 1\leq k\leq n_2,\, |\alpha|=\varkappa}$ by X.

- Let us denote by d the control distance associated with the Grushin operator.

- Let us denote by d the control distance associated with the Grushin operator.
- Let $B(x,r) := \{y \in \mathbb{R}^{n_1+n_2} : d(x,y) < r\}$ and $|B(x,r)|$ denotes the (Lebesgue) measure of the ball $B(x,r)$. Then

$$|B(x,r)| \sim \ r^{n_1+n_2} \max\{r, |x'|\}^{\varkappa n_2},$$

for all $x \in \mathbb{R}^{n_1+n_2}$ and $r > 0$.

- Let us denote by d the control distance associated with the Grushin operator.

- Let $B(x,r) := \{y \in \mathbb{R}^{n_1+n_2} : d(x,y) < r\}$ and $|B(x,r)|$ denotes the (Lebesgue) measure of the ball $B(x,r)$. Then

$$|B(x,r)| \sim \; r^{n_1+n_2} \max\{r, |x'|\}^{\varkappa n_2},$$

for all $x \in \mathbb{R}^{n_1+n_2}$ and $r > 0$.

- The above implies that the space $(\mathbb{R}^{n_1+n_2}, d, |\cdot|)$ is a homogeneous metric space, that is, the underlying measure satisfies the following doubling condition

$$|B(x,sr)| \lesssim_{n_1,n_2,\kappa} (1+s)^Q |B(x,r)|$$

for $s > 0$, where Q denotes the homogeneous dimension $n_1 + (1+\varkappa)n_2$.

Spectral decomposition of Grushin operator

- For Schwartz class function f, we can write

$$G_{\varkappa}f(x) = \int_{\mathbb{R}^{n_2}} e^{-i\lambda\cdot x''} H_{\varkappa}(\lambda) f^{\lambda}(x')\, d\lambda,$$

where $H_{\varkappa}(\lambda) = -\Delta_{x'} + |\lambda|^2|x'|^{2\varkappa}$ and $f^{\lambda}(x') = \int_{\mathbb{R}} f(x', x'') e^{i\lambda\cdot x''}\, dx''$.

Spectral decomposition of Grushin operator

- For Schwartz class function f, we can write

$$G_\varkappa f(x) = \int_{\mathbb{R}^{n_2}} e^{-i\lambda \cdot x''} H_\varkappa(\lambda) f^\lambda(x')\, d\lambda,$$

where $H_\varkappa(\lambda) = -\Delta_{x'} + |\lambda|^2 |x'|^{2\varkappa}$ and $f^\lambda(x') = \int_{\mathbb{R}} f(x', x'') e^{i\lambda \cdot x''}\, dx''$.

- Using the spectral decomposition of the operator $H_\varkappa(\lambda)$, one can write the the spectral decomposition of the Grushin operator $G_\varkappa$ as follows:

$$G_\varkappa f(x) = \int_{\mathbb{R}^{n_2}} e^{-i\lambda \cdot x''} \sum_{k \in \mathbb{N}} |\lambda|^{\frac{2}{\varkappa+1}} \nu_{\varkappa,k} P_{\varkappa,k}(\lambda) f^\lambda(x')\, d\lambda.$$

Spectral decomposition of Grushin operator

- For Schwartz class function f, we can write

$$G_{\varkappa}f(x) = \int_{\mathbb{R}^{n_2}} e^{-i\lambda\cdot x''} H_{\varkappa}(\lambda) f^{\lambda}(x')\, d\lambda,$$

where $H_{\varkappa}(\lambda) = -\Delta_{x'} + |\lambda|^2 |x'|^{2\varkappa}$ and $f^{\lambda}(x') = \int_{\mathbb{R}} f(x', x'') e^{i\lambda\cdot x''}\, dx''$.

- Using the spectral decomposition of the operator $H_{\varkappa}(\lambda)$, one can write the the spectral decomposition of the Grushin operator $G_{\varkappa}$ as follows:

$$G_{\varkappa}f(x) = \int_{\mathbb{R}^{n_2}} e^{-i\lambda\cdot x''} \sum_{k\in\mathbb{N}} |\lambda|^{\frac{2}{\varkappa+1}} \nu_{\varkappa,k} P_{\varkappa,k}(\lambda) f^{\lambda}(x')\, d\lambda.$$

- Given a bounded measurable function m on $\mathbb{R}^{n_1+n_2} \times \mathbb{R}_+$, we consider the Grushin pseudo-multiplier $m(\cdot, G_{\varkappa})$, defined by

$$m(x, G_{\varkappa}) f(x) := \int_{\mathbb{R}^{n_2}} e^{-i\lambda\cdot x''} \sum_{k\in\mathbb{N}} m\left(x, |\lambda|^{\frac{2}{\varkappa+1}} \nu_{\varkappa,k}\right) P_{\varkappa,k}(\lambda) f^{\lambda}(x')\, d\lambda,$$

for Schwartz class functions f on $\mathbb{R}^{n_1+n_2}$.

Symbol class for Grushin pseudo-multipliers

Definition

For any $\sigma \in \mathbb{R}$ and $\rho, \delta \geq 0$, define the symbol class $\mathcal{S}^{\sigma}_{\rho,\delta}(G_{\varkappa})$ to be the set of all functions $m \in C^{\infty}\left(\mathbb{R}^{n_1+n_2} \times \mathbb{R}_{+}\right)$ which satisfy the following estimate:

$$\left| X^{\Gamma} \partial_{\eta}^{l} m(x, \eta) \right| \leq_{\Gamma, l} (1+\eta)^{\frac{\sigma}{2} - (1+\rho)\frac{l}{2} + \delta \frac{|\Gamma|}{2}}$$

for all $\Gamma \in \mathbb{N}^{n_0}$ and $l \in \mathbb{N}$. Here $n_0 = n_1 + n_2 \binom{\varkappa + n_1 - 1}{n_1 - 1}$.

Symbol class for Grushin pseudo-multipliers

Definition

For any $\sigma \in \mathbb{R}$ and $\rho, \delta \geq 0$, define the symbol class $\mathcal{S}^{\sigma}_{\rho,\delta}(G_{\varkappa})$ to be the set of all functions $m \in C^{\infty}(\mathbb{R}^{n_1+n_2} \times \mathbb{R}_+)$ which satisfy the following estimate:

$$\left| X^{\Gamma} \partial_{\eta}^{l} m(x,\eta) \right| \leq_{\Gamma,l} (1+\eta)^{\frac{\sigma}{2}-(1+\rho)\frac{l}{2}+\delta\frac{|\Gamma|}{2}}$$

for all $\Gamma \in \mathbb{N}^{n_0}$ and $l \in \mathbb{N}$. Here $n_0 = n_1 + n_2\binom{\varkappa+n_1-1}{n_1-1}$.

Recently, Bagchi and Garg studied the Grushin pseudo- multipliers for the case $\varkappa = 1$. They proved an analogue of Calderón-Vaillancourt type theorem for the Grushin pseudo-multipliers.

Symbol class for Grushin pseudo-multipliers

Definition

For any $\sigma \in \mathbb{R}$ and $\rho, \delta \geq 0$, define the symbol class $\mathcal{S}^{\sigma}_{\rho,\delta}(G_{\varkappa})$ to be the set of all functions $m \in C^{\infty}\left(\mathbb{R}^{n_1+n_2} \times \mathbb{R}_+\right)$ which satisfy the following estimate:

$$\left| X^{\Gamma} \partial^{l}_{\eta} m(x, \eta) \right| \leq_{\Gamma, l} (1+\eta)^{\frac{\sigma}{2} - (1+\rho)\frac{l}{2} + \delta \frac{|\Gamma|}{2}}$$

for all $\Gamma \in \mathbb{N}^{n_0}$ and $l \in \mathbb{N}$. Here $n_0 = n_1 + n_2 \binom{\varkappa + n_1 - 1}{n_1 - 1}$.

Recently, Bagchi and Garg studied the Grushin pseudo- multipliers for the case $\varkappa = 1$. They proved an analogue of Calderón-Vaillancourt type theorem for the Grushin pseudo-multipliers.

Theorem (Bagchi-Garg, 2021)

Let $m \in \mathcal{S}^{0}_{\rho,\delta}(G)$ for some $0 \leq \delta < \rho \leq 1$. Then the operator $m(x, G)$ extends to a bounded on $L^2(\mathbb{R}^{n_1+n_2})$ to itself.

Main result

Theorem (Bagchi-B.-Garg-Ghosh, 2021)

Let $m : \mathbb{R}^{n_1+n_2} \times \mathbb{R}_+ \to \mathbb{C}$ *be such that for all* $0 \leq l \leq Q+1$

$$\left|\partial_\tau^l m(x,\tau)\right| \lesssim_{\Gamma,l} (1+\tau)^{-l}.$$

Assume also that the pseudo-multiplier operator $m(x, G_\varkappa)$ *is bounded on* $L^2(\mathbb{R}^{n_1+n_2})$*. Then* $m(x, G_\varkappa)$ *is of weak type* $(1,1)$ *and as a consequence* $m(x, G_\varkappa) : L^p(\mathbb{R}^{n_1+n_2}) \to L^p(\mathbb{R}^{n_1+n_2})$ *is bounded for* $1 < p < 2$*.*

Sparse operator

- Let $\mathcal{S}$ be a family of dyadic cubes.

Definition

We say a family of sets $S \subset \mathcal{S}$ is η-sparse, $0 < \eta < 1$, if for every $\mathcal{Q} \in S$ there exists a set $E_{\mathcal{Q}} \subset \mathcal{Q}$ such that $|E_{\mathcal{Q}}| \geq \eta|\mathcal{Q}|$ and the sets $\{E_{\mathcal{Q}}\}_{\mathcal{Q} \in S}$ are pairwise disjoint.

Sparse operator

- Let $\mathcal{S}$ be a family of dyadic cubes.

Definition

We say a family of sets $S \subset \mathcal{S}$ is η-sparse, $0 < \eta < 1$, if for every $\mathcal{Q} \in S$ there exists a set $E_{\mathcal{Q}} \subset \mathcal{Q}$ such that $|E_{\mathcal{Q}}| \geq \eta|\mathcal{Q}|$ and the sets $\{E_{\mathcal{Q}}\}_{\mathcal{Q}\in S}$ are pairwise disjoint.

- For a sparse family S and $1 \leq r < \infty$, we consider the sparse operator defined as following

$$\mathcal{A}_{r,S}f(x) = \sum_{\mathcal{Q}\in S} \left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |f|^r\right)^{\frac{1}{r}} \chi_{\mathcal{Q}}(x).$$

Sparse operator

- Let $\mathcal{S}$ be a family of dyadic cubes.

Definition

We say a family of sets $S \subset \mathcal{S}$ is η-sparse, $0 < \eta < 1$, if for every $\mathcal{Q} \in S$ there exists a set $E_{\mathcal{Q}} \subset \mathcal{Q}$ such that $|E_{\mathcal{Q}}| \geq \eta|\mathcal{Q}|$ and the sets $\{E_{\mathcal{Q}}\}_{\mathcal{Q}\in S}$ are pairwise disjoint.

- For a sparse family S and $1 \leq r < \infty$, we consider the sparse operator defined as following

$$\mathcal{A}_{r,S}f(x) = \sum_{\mathcal{Q}\in S} \left(\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |f|^r \right)^{\frac{1}{r}} \chi_{\mathcal{Q}}(x).$$

- Then for $r < p < \infty$, $w \in A_{p/r}(\mathbb{R}^{n_1+n_2})$ and $f \in L^p(\mathbb{R}^{n_1+n_2}, w)$ we have

$$\|\mathcal{A}_{r,S}f\|_{L^p(w)} \lesssim [w]_{A_p(\mathbb{R}^{n_1+n_2})}^{\max\{\frac{1}{p-r},1\}} \|f\|_{L^p(w)}.$$

Sparse domination result

- For any sublinear operator T, the grand maximal truncated operator $\mathcal{M}^{\#}_{T,s}$ is defined by

$$\mathcal{M}^{\#}_{T,s}f(x) = \sup_{B\in\mathcal{S}:\ B\ni x}\ \sup_{y,z\in B} |T(f\chi_{\mathbb{R}^{n_1+n_2}\setminus sB})(y) - T(f\chi_{\mathbb{R}^{n_1+n_2}\setminus sB})(z)|.$$

Sparse domination result

- For any sublinear operator T, the grand maximal truncated operator $\mathcal{M}^{\#}_{T,s}$ is defined by

$$\mathcal{M}^{\#}_{T,s}f(x) = \sup_{B\in\mathcal{S}:\ B\ni x}\ \sup_{y,z\in B} |T(f\chi_{\mathbb{R}^{n_1+n_2}\setminus sB})(y) - T(f\chi_{\mathbb{R}^{n_1+n_2}\setminus sB})(z)|.$$

Theorem (Lerner-Ombrosi, 2020; Lorist, 2021)

Let T be a sublinear operator of weak-type (p,p) and $\mathcal{M}^{\#}_{T,\alpha}$ is weak type (q,q), where $1 \leq p, q < \infty$ and $s \geq \frac{3c_d^2}{\delta}$. Let $r = \max\{p,q\}$. Then for every compactly supported bounded measurable function f, there exist a η-sparse family $S \subset \mathcal{S}$ such that

$$|Tf(x)| \leq C\mathcal{A}_{r,S}f(x)$$

for almost every x.

Weighted boundedness results

Theorem (Bagchi-B.-Garg-Ghosh, 2021)

For a fixed $0 \leq \delta < 1$, *let* $m \in L^{\infty}(\mathbb{R}^{n_1+n_2} \times \mathbb{R}_+)$ *be such that*

$$\left|\partial_\eta^l m(x,\eta)\right| \leq_l (1+\eta)^{-l}, \quad \text{for all} \quad l \leq \lfloor Q/2 \rfloor + 1,$$

$$\text{and} \quad \left|X_x \partial_\eta^l m(x,\eta)\right| \leq_{l,\delta} (1+\eta)^{-l+\frac{\delta}{2}}, \quad \text{for all} \quad l \leq \lfloor Q/2 \rfloor.$$

Assume also that the operator $T = m(x, G_\varkappa)$ *is bounded on* $L^2(\mathbb{R}^{n_1+n_2})$. *Then, for every compactly supported bounded measurable function* f *there exists a sparse family* $S \subset \mathcal{S}$ *such that*

$$|Tf(x)| \lesssim_T \mathcal{A}_{2,S} f(x),$$

for almost every $x \in \mathbb{R}^{n_1+n_2}$.

Weighted boundedness results

Theorem (Bagchi-B.-Garg-Ghosh, 2021)

For a fixed $0 \leq \delta < 1$, *let* $m \in L^\infty\left(\mathbb{R}^{n_1+n_2} \times \mathbb{R}_+\right)$ *be such that*

$$\left|\partial_\eta^l m(x,\eta)\right| \leq_l (1+\eta)^{-l}, \quad \text{for all} \quad l \leq Q+1,$$
$$\text{and} \quad \left|X_x \partial_\eta^l m(x,\eta)\right| \leq_{l,\delta} (1+\eta)^{-l+\frac{\delta}{2}}, \quad \text{for all} \quad l \leq Q.$$

Assume also that the operator T *is bounded on* $L^2(\mathbb{R}^{n_1+n_2})$. *Then, for every compactly supported bounded measurable function* f *there exists a sparse family* $S \subset \mathcal{S}$ *such that*

$$|Tf(x)| \lesssim_T \mathcal{A}_{1,S} f(x),$$

for almost every $x \in \mathbb{R}^{n_1+n_2}$.

General operator T

- Choose and fix $\psi_0 \in C_c^\infty((-2,2))$ and $\psi_1 \in C_c^\infty((1/2,2))$ such that for all $\eta \geq 0$ we have $0 \leq \psi_0(\eta), \psi_1(\eta) \leq 1$, and

$$\sum_{j=0}^{\infty} \psi_j(\eta) = 1$$

where $\psi_j(\eta) = \psi_1\left(2^{-(j-1)}\eta\right)$ for $j \geq 2$.

General operator T

- Choose and fix $\psi_0 \in C_c^\infty((-2,2))$ and $\psi_1 \in C_c^\infty((1/2,2))$ such that for all $\eta \geq 0$ we have $0 \leq \psi_0(\eta), \psi_1(\eta) \leq 1$, and

$$\sum_{j=0}^{\infty} \psi_j(\eta) = 1$$

  where $\psi_j(\eta) = \psi_1\left(2^{-(j-1)}\eta\right)$ for $j \geq 2$.

- Given $T \in \mathcal{B}\left(L^2(\mathbb{R}^{n_1+n_2})\right)$, we break it into a countable sum of operators as follows. For each $j \in \mathbb{N}$, let us define

$$T_j = TS_j$$

  where S_j's are the Grushin multiplier operators $S_j = \psi_j(G_\varkappa)$. Then, we have $T = \sum_j T_j$ with convergence in the strong operator topology of $\mathcal{B}\left(L^2(\mathbb{R}^{n_1+n_2})\right)$.

L^2-conditions on the kernel

We denote the kernel of the operator T_j by $T_j(x, y)$.

L^2-conditions on the kernel

We denote the kernel of the operator T_j by $T_j(x, y)$.

There exists some $R_0 \in (0, \infty)$ such that for all $\mathfrak{r} \in [0, R_0]$ and for every compact set $\Lambda \subset \mathbb{R}^{n_1+n_2}$ we have

1. $$\sup_{x \in \mathbb{R}^{n_1+n_2}} |B(x, 2^{-j/2})| \int_{\mathbb{R}^{n_1+n_2}} d(x, y)^{2\mathfrak{r}} |T_j(x, y)|^2 \, dy \lesssim_{R_0} 2^{-j\mathfrak{r}},$$

L^2-conditions on the kernel

We denote the kernel of the operator T_j by $T_j(x,y)$.

There exists some $R_0 \in (0,\infty)$ such that for all $\mathfrak{r} \in [0, R_0]$ and for every compact set $\Lambda \subset \mathbb{R}^{n_1+n_2}$ we have

1. $$\sup_{x \in \mathbb{R}^{n_1+n_2}} |B(x, 2^{-j/2})| \int_{\mathbb{R}^{n_1+n_2}} d(x,y)^{2\mathfrak{r}} |T_j(x,y)|^2 \, dy \lesssim_{R_0} 2^{-j\mathfrak{r}},$$

2. $$\sup_{x \in \mathbb{R}^{n_1+n_2}} |B(x, 2^{-j/2})| \int_{\Lambda} d(x,y)^{2\mathfrak{r}} |X_x T_j(x,y)|^2 \, dy \lesssim_{R_0,\Lambda} 2^{-j\mathfrak{r}} 2^j.$$

Theorem (Bagchi-B.-Garg-Ghosh, 2021)

Let $T \in \mathcal{B}\left(L^2(\mathbb{R}^{n_1+n_2})\right)$. Suppose that the integral kernels $T_j(x,y)$ satisfy conditions (1) and (2) for some $R_0 > Q/2$. Then, we have the following pointwise almost everywhere estimate

$$\mathcal{M}^{\#}_{T,s} f(x) \lesssim_{T,s} \mathcal{M}_2 f(x),$$

for every bounded measurable function f with compact support.

Lemma (Bagchi-B.-Garg-Ghosh, 2021)

For $2 \leq p \leq \infty$ and every $\mathfrak{r} > 0$ and $\epsilon > 0$, we have

$$|B(x, R^{-1})|^{1/2} \left\| |B(\cdot, R^{-1})|^{1/2-1/p} (1 + Rd(x, \cdot))^{\mathfrak{r}} X_x^{\Gamma} K_{m(x, G_{\varkappa})}(x, \cdot) \right\|_p$$
$$\lesssim_{\Gamma, p, \mathfrak{r}, \epsilon} \sup_{x_0} \sum_{\Gamma_1 + \Gamma_2 = \Gamma} R^{|\Gamma_1|} \| X_x^{\Gamma_2} m(x_0, R^2 \cdot) \|_{W^{\infty}_{\mathfrak{r}+\epsilon}},$$

for all $\Gamma \in \mathbb{N}^{n_0}$ and for every bounded Borel function $m : \mathbb{R}^{n_1+n_2} \times \mathbb{R} \to \mathbb{C}$ whose support in the last variable is in $[0, R^2]$ for any $R > 0$.

Reference

- S. Bagchi and R. Garg, *On L^2-boundedness of pseudo-multipliers associated to the Hermite and Grushin operators,* Preprint.
- S. Bagchi and S. Thangavelu, *On Hermite pseudo-multipliers,* J. Funct. Anal. **268** (2015), no. 1, 140–170.
- D. Beltran and L. Cladek, *Sparse bounds for pseudodifferential operators*, J. Anal. Math. **140** (2020), no. 1, 89–116.
- A-P. Calderón and R. Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan **23** (1971), 374–378.
- X. T. Duong, E. M. Ouhabaz, and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal. **196** (2002), no. 2, 443–485.

Reference

J. Epperson, *Hermite multipliers and pseudo-multipliers*, Proc. Amer. Math. Soc. **124** (7) (1996), 2061–2068.

V. V. Grushin, *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.) **83** (1970), 456–473.

E. Lorist, *On pointwise l^r-sparse domination in a space of homogeneous type*, J. Geom. Anal. **31** (2021), no. 9, 9366–9405.

N. Miller, *Weighted Sobolev spaces and pseudodifferential operators with smooth symbols,* Trans. Amer. Math. Soc. **269** (1982), no. 1, 91–109.

E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals,* Princeton University Press, Princeton (1993).

Thank You!