On Some Analogues of Beurling's Theorem

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Uncertainty principle

- For $f \in L^1(\mathbb{R})$, we define the Fourier transform of f by

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) e^{-ixy}\, dx, \quad \text{for all } y \in \mathbb{R}.$$

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- An example

Theorem

Let $f \in L^1(\mathbb{R})$ satisfying

$$|\widehat{f}(y)| \leq Ce^{-a|y|} \quad \text{for all } y \in \mathbb{R} \text{ for some } a > 0.$$

If f vanishes on any non-empty open subset of $\mathbb{R}$, then f is identically zero.

Levinson's uncertainty principle

Theorem (Levinson, 1936)

Let $f \in L^1(\mathbb{R})$ *and* $\psi : [0, \infty) \to [0, \infty)$ *increasing function satisfying*

$$\int_0^\infty \frac{\psi(\xi)}{1+\xi^2} d\xi = \infty, \tag{1}$$

and

$$|\widehat{f}(\xi)| \leq Ce^{-\psi(|\xi|)}, \quad \text{for almost every } \xi \in \mathbb{R}. \tag{2}$$

If f *vanishes on any non-empty open set in* $\mathbb{R}$, *then* f *is identically zero.*

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- We note that as ψ is increasing, from equation (1), we have $\psi(x) \uparrow \infty$ as $x \to \infty$. So from equation (2) it follows that $\widehat{f}$ decays to zero.

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- We note that as ψ is increasing, from equation (1), we have $\psi(x) \uparrow \infty$ as $x \to \infty$. So from equation (2) it follows that $\widehat{f}$ decays to zero.
- Levinson actually worked with a more general estimate of the form $\int_{\mathbb{R}} |\widehat{f}(\xi)| e^{\psi(|\xi|)} d\xi < \infty$ instead of (2).

Beurling's uncertainty principle

Beurling improved the result and replaced open set with set of positive Lebesgue measure. He proved the result for complex Borel measure μ on $\mathbb{R}$. We define the Fourier transform $\widehat{\mu}$ of μ by

$$\widehat{\mu}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} d\mu(t), \text{ for } \lambda \in \mathbb{R}.$$

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$$\widehat{\mu}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} d\mu(t), \text{ for } \lambda \in \mathbb{R}.$$

Theorem (Beurling, 1989)

Let μ be a complex Borel measure on $\mathbb{R}$ and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function satisfying

$$\int_0^\infty \frac{\psi(x)}{1 + x^2} dx = \infty,$$

$$\int_{\mathbb{R}} e^{\psi(|x|)} |d\mu(x)| < \infty.$$

If $\widehat{\mu}$ vanishes on $\Lambda \subset \mathbb{R}$ such that $m(\overline{\Lambda}) > 0$ then μ is identically 0.

- Beurling assumed the estimate

$$\int_0^\infty \frac{1}{1+x^2} \log\left(\frac{1}{\int_x^\infty |d\mu(t)|}\right) dx = \infty,$$

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- It was proved as a consequence of a characterization of the Beurling quasianalytic class (a generalisation of the famous Denjoy-Carleman quasianalytic class and Bernstein quasianalytic class), which used the concept of harmonic measure.

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- It was proved as a consequence of a characterization of the Beurling quasianalytic class (a generalisation of the famous Denjoy-Carleman quasianalytic class and Bernstein quasianalytic class), which used the concept of harmonic measure.
- In the book of Koosis, it was shown that the theorem of Levinson can also be obtained as a consequence of completeness of linear span of exponentials in certain weighted space of continuous functions. This leads us to another problem in harmonic analysis which is of independent interest.

Weighted Approximation of Exponentials

- First we define a weighted space of continuous functions. Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(x) \to \infty$ as $x \to \infty$. Consider

$$C_\psi(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is continuous and } \lim_{|x| \to \infty} \frac{f(x)}{e^{\psi(|x|)}} = 0 \right\}.$$

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- We define a norm on $C_\psi(\mathbb{R}^n)$ by

$$\|f\|_\psi = \sup_{x\in\mathbb{R}^n} \frac{|f(x)|}{e^{\psi(|x|)}}, \quad \text{for } f \in C_\psi(\mathbb{R}^n).$$

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- It is easy to see that $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$ is a normed linear space.
- For any $\lambda \in \mathbb{R}^n$, we define the continuous functions $e_\lambda : \mathbb{R}^n \to \mathbb{C}$ by

$$e_\lambda(x) = e^{-i\lambda \cdot x} \quad \text{for } x \in \mathbb{R}^n.$$

- For any $\Lambda \subset \mathbb{R}^n$, we consider the linear span of the above functions given by

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- Problems regarding weighted approximation of exponentials pose the question for which type of sets $\Lambda \subset \mathbb{R}^n$, $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$.

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- Problems regarding weighted approximation of exponentials pose the question for which type of sets $\Lambda \subset \mathbb{R}^n$, $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$.
- For example, when $\Lambda \subset \mathbb{R}^n$ is a non-trivial open set then $\Phi_\Lambda(\mathbb{R}^n)$ is always dense in $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$. This result was used in the alternative proof of Levinson's theorem.

- To get the result regarding exponential density from Beurling's theorem, we first need to explicitly find the dual of $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$, denoted by $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)^*$, the space of all bounded linear functionals on $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$. For that we required to assume that ψ is continuous.

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Lemma

$(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$ is a Banach space isometrically isomorphic to $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ and its dual is given by

$$(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)^* = \left\{\beta \in \mathcal{M}(\mathbb{R}^n) : \int_{\mathbb{R}^n} e^{\psi(|x|)} |d\beta(x)| < \infty\right\}.$$

($\mathcal{M}(\mathbb{R}^n)$ is the space of all complex Borel measure $\mathbb{R}^n$.)

Theorem

Let $\psi : [0,\infty) \to [0,\infty)$ *be a continuous, increasing function satisfying*

$$\int_0^\infty \frac{\psi(x)}{1+x^2}\, dx = \infty.$$

For $\Lambda \subset \mathbb{R}$ *such that* $m(\overline{\Lambda}) > 0$, $\Phi_\Lambda(\mathbb{R})$ *is dense in* $(C_\psi(\mathbb{R}), \|\cdot\|_\psi)$.

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Proof.

- Let $T \in (C_\psi(\mathbb{R}), \| \cdot \|_\psi)^*$ such that T vanishes on $\Phi_\Lambda(\mathbb{R})$.

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Proof.

- Let $T \in (C_\psi(\mathbb{R}), \|\cdot\|_\psi)^*$ such that T vanishes on $\Phi_\Lambda(\mathbb{R})$.
- $$T(f) = \int_{\mathbb{R}} f(t)\, d\beta(t), \qquad \text{for } f \in C_\psi(\mathbb{R}),$$

  where $\beta \in \mathcal{M}(\mathbb{R}^n)$ satisfies $\displaystyle\int_{\mathbb{R}} e^{\psi(|x|)}|d\beta(x)| < \infty$.

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  where $\beta \in \mathcal{M}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}} e^{\psi(|x|)}|d\beta(x)| < \infty.$
- T vanishes on $\Phi_\Lambda(\mathbb{R})$ implies
  $$\int_{\mathbb{R}} e^{-i\lambda t} d\beta(t) = 0, \quad \forall\ \lambda \in \Lambda.$$ □

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- An open set $U \subseteq \mathbb{R}^n$ always contains a set of the form $U_1 \times U_2 \times \cdots \times U_n$ where each $U_j \subseteq \mathbb{R}$ is open in $\mathbb{R}$ for $1 \leq j \leq n$. This type of property of the set is an important tool in the technique of extending the result to $\mathbb{R}^n$ using the result of $\mathbb{R}$. However, this property does not hold for sets of positive Lebesgue measure in $\mathbb{R}^n$.

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- It is due to this property of sets of positive Lebesgue measure in $\mathbb{R}^n$ that we have assumed sets of the special form instead of any set of positive Lebesgue measure in $\mathbb{R}^n$.
- We call $\Lambda \subset \mathbb{R}^n$ to be "a set of positive rectangle type" if Λ contains a set of the form $\Lambda_1 \times \cdots \times \Lambda_n$, where $\Lambda_j \subset \mathbb{R}$ for each $1 \leq j \leq n$ such that the closure of Λ_j has positive Lebesgue measure in $\mathbb{R}$, that is, $m(\overline{\Lambda_j}) > 0$.

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Theorem

Let ψ be a non-negative, continuous, increasing function on $[0,\infty)$ such that $\psi(x)\to\infty$ as $x\to\infty$. The space $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $(C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)$ for any positive rectangle type set $\Lambda\subset\mathbb{R}^n$ if and only if

$$\int_0^\infty \frac{\psi(x)}{1+x^2}\,dx=\infty. \tag{3}$$

Theorem

Let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(x) \to \infty$ as $x \to \infty$ and $I = \int_0^\infty \frac{\psi(x)}{1+x^2}dx$.

(a) *Let μ be a complex Borel measure on $\mathbb{R}^n$ satisfying*

$$\int_{\mathbb{R}^n} e^{\psi(|x|)}|d\mu(x)| < \infty. \tag{4}$$

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If $\widehat{\mu}$ *vanishes on a set* $\Lambda \subset \mathbb{R}^n$ *of positive rectangle type and* $I = \infty$*, then* μ *is identically zero.*

(b) *If* $I < \infty$*, then there exists* $\mu \in \mathcal{M}(\mathbb{R}^n)$ *satisfying (4) such that* $\widehat{\mu}$ *vanishes on a set of positive rectangle type.*

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(b) *If $I < \infty$, then there exists $\mu \in \mathcal{M}(\mathbb{R}^n)$ satisfying (4) such that $\widehat{\mu}$ vanishes on a set of positive rectangle type.*

Proof.

- If $T_\mu(f) = \int_{\mathbb{R}^n} f(x) d\mu(x)$, (4) implies, $T_\mu \in (C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)^*$.

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Proof.

- If $T_\mu(f) = \int_{\mathbb{R}^n} f(x) d\mu(x)$, (4) implies, $T_\mu \in (C_\psi(\mathbb{R}^n), \|\cdot\|_\psi)^*$.
- $\widehat{\mu}$ vanishes on a set $\Lambda \subset \mathbb{R}^n$ implies T_μ vanishes on $\Phi_\Lambda(\mathbb{R}^n)$.

Theorem

Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $\psi(x) \to \infty$ as $x \to \infty$. Then the following are equivalent:

(1) $\displaystyle\int_0^\infty \frac{\psi(x)}{1+x^2}\,dx = \infty.$

(2) *$\Phi_\Lambda(\mathbb{R}^n)$ is dense in $(C_\psi(\mathbb{R}^n), |\cdot\|_\psi)$, for any positive rectangle type set $\Lambda \subset \mathbb{R}^n$.*

(3) *There does not exist a non-zero complex Borel measure μ on $\mathbb{R}^n$ such that*
$$\int_{\mathbb{R}^n} e^{\psi(|x|)}|d\mu(x)| < \infty,$$
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(2) $\Phi_\Lambda(\mathbb{R}^n)$ *is dense in* $(C_\psi(\mathbb{R}^n), |\cdot\|_\psi)$, *for any positive rectangle type set* $\Lambda \subset \mathbb{R}^n$.

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$$\int_{\mathbb{R}^n} e^{\psi(|x|)} |d\mu(x)| < \infty,$$

and $\widehat{\mu}$ *vanishes on a set* $\Lambda \subset \mathbb{R}^n$ *of positive rectangle type.*

$$\begin{array}{ccc} \text{Beurling's result on } \mathbb{R} & \longrightarrow & \text{Exponential density on } \mathbb{R} \\ \big\uparrow & & \big\downarrow \\ \text{Beurling's result on } \mathbb{R}^n & \longleftarrow & \text{Exponential density on } \mathbb{R}^n \end{array}$$

Another Version Weighted Approximation of Exponentials

- In another version, one studies conditions on $\Lambda \subset \mathbb{R}^n$ and μ that ensure completeness, that is, density of $\Phi_\Lambda(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n, \mu)$.

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Theorem

Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\int_0^\infty \frac{\psi(x)}{1 + x^2} dx = \infty \tag{5}$$

and μ be a positive measure satisfying

$$\int_{\mathbb{R}^n} e^{\psi(|x|)} d\mu(x) < \infty. \tag{6}$$

For any set of positive rectangle type $\Lambda \subset \mathbb{R}^n$, the space $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \mu)$, $1 \leq p < \infty$.

Proof.

- Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \mu)$, for $1 \leq p < \infty$ it is enough to prove that $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ with respect to the corresponding $\|\cdot\|_p$ norm of $L^p(\mathbb{R}^n, \mu)$, for $1 \leq p < \infty$.

Proof.

- Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \mu)$, for $1 \leq p < \infty$ it is enough to prove that $\Phi_\Lambda(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R}^n)$ with respect to the corresponding $\| \cdot \|_p$ norm of $L^p(\mathbb{R}^n, \mu)$, for $1 \leq p < \infty$.
- Define $\psi_1(x) = \dfrac{\psi(x)}{p}$, which satisfies $\displaystyle\int_0^\infty \frac{\psi_1(x)}{1+x^2} dx = \infty$.

Proof.

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- Define $\psi_1(x) = \dfrac{\psi(x)}{p}$, which satisfies $\displaystyle\int_0^\infty \frac{\psi_1(x)}{1+x^2}dx = \infty$.
- Consider $f \in C_c(\mathbb{R}^n) \subseteq C_{\psi_1}(\mathbb{R}^n)$. Given any $\epsilon > 0$, we get $g \in \Phi_\Lambda(\mathbb{R}^n)$ such that $\|f - g\|_{\psi_1} < \epsilon$. From (6) we get $C > 0$ such that

$$\begin{aligned} \|f-g\|_p^p &= \int_{\mathbb{R}^n} \frac{|f(x)-g(x)|^p}{e^{p\psi_1(|x|)}} e^{\psi(|x|)} d\mu(x) \\ &\leq \|f-g\|_{\psi_1}^p \int_{\mathbb{R}^n} e^{\psi(|x|)} d\mu(x) \; < \; C\epsilon^p. \end{aligned}$$

□

A Generalisation of Beurling's theorem

For $f \in L^1(\mathbb{R}^n, \mu)$, we define its Fourier transform by

$$\mathcal{F}_\mu(\lambda) = \int_{\mathbb{R}^n} f(x) e^{-i\lambda x} d\mu(x).$$

Theorem

Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\int_0^\infty \frac{\psi(x)}{1 + x^2} dx = \infty$$

and μ be a positive measure satisfying

$$\int_{\mathbb{R}^n} e^{\psi(|x|)} d\mu(x) < \infty.$$

If $f \in L^p(\mathbb{R}^n, \mu)$, for $1 < p \leq \infty$, is such that $\mathcal{F}_\mu(f)$ vanishes on a set of positive rectangle type, then f is zero a.e. μ.

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- For any fixed $z \in \mathcal{D}$, there is a unique measure $\omega_{\mathcal{D}}(\cdot, z)$ on $\partial\mathcal{D}$ determined by $U_\phi(z)$ in the following way

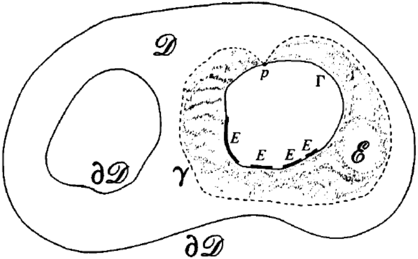
$$U_\phi(z) = \int_{\partial\mathcal{D}} \phi(\zeta)\, d\omega_{\mathcal{D}}(\zeta, z).$$

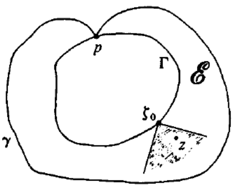
This positive Radon measure $\omega_{\mathcal{D}}(\cdot, z)$ on $\partial\mathcal{D}$, of total mass 1 is called the harmonic measure relative to $\mathcal{D}$ as seen from z.

Theorem

Let E be a closed set lying on a single component Γ of $\partial\mathcal{D}$ and $\mathcal{E} \subseteq \mathcal{D}$ be a simply connected domain whose boundary contains Γ. For almost every $\zeta_0 \in E$, if $z \in \mathcal{E}$ tends to ζ_0 from within an acute angle with vertex at ζ_0, lying strictly in $\mathcal{E}$, (henceforth denoted by $z \twoheadrightarrow \zeta_0$), then

$$\omega_{\mathcal{D}}(E, z) \longrightarrow 1.$$





Another theorem regarding Harmonic measure

Theorem (Theorem on two constants)

Let f be a function which is analytic, bounded in $\mathcal{D}$ and continuous up to $\partial\mathcal{D}$. If $|f(\zeta)| \leq M$ for $\zeta \in \partial\mathcal{D}$, and there is a Borel set $E \subseteq \partial\mathcal{D}$ with $|f(\zeta)| \leq m$ $(< M)$ for $\zeta \in E$, then

$$|f(z)| \leq m^{\omega_{\mathcal{D}}(E,z)} M^{1-\omega_{\mathcal{D}}(E,z)}, \qquad \text{for } z \in \mathcal{D}.$$

Preliminaries for Hankel transform

- For $\alpha > -\frac{1}{2}$, we define the measure $d\gamma_\alpha$ on $[0, \infty)$ by
$$d\gamma_\alpha(\lambda) = \frac{\lambda^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)}\, d\lambda.$$

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- For $f \in L^1_\alpha(\mathbb{R}_+) = L^1([0,\infty), d\gamma_\alpha)$, we define its Hankel transform of order $\alpha > -\frac{1}{2}$ by

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- If $f \in L^1_\alpha(\mathbb{R}_+)$ such that $\mathcal{H}_\alpha f \in L^1_\alpha(\mathbb{R}_+)$, then
$$f(\lambda) = \int_0^\infty \mathcal{H}_\alpha f(t) j_\alpha(\lambda t)\, d\gamma_\alpha(t), \quad \text{for almost every } \lambda \in [0, \infty).$$

Beurling's theorem for Hankel transform

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Theorem

Let $\alpha > -\frac{1}{2}$, $\psi : [0, \infty) \to [0, \infty)$ *be an increasing function such that* $\psi(x) \to \infty$ *as* $x \to \infty$ *and consider* $I = \int_0^\infty \frac{\psi(x)}{1+x^2}\, dx$.

(a) *Let* $I = \infty$ *and* $f \in L^1_\alpha(\mathbb{R}_+)$ *satisfying*

$$\int_0^\infty |\mathcal{H}_\alpha(f)(t)| e^{\psi(t)}\, d\gamma_\alpha(t) < \infty. \tag{7}$$

If f *vanishes on a set* $E \subseteq [0, \infty)$ *of positive Lebesgue measure, then* $f = 0$ *almost everywhere on* $[0, \infty)$.

(b) *If* I *is finite then there exists a non-trivial function* f *on* $\mathbb{R}_+$ *satisfying (7) which vanishes on a set of positive Lebesgue measure in* $\mathbb{R}_+$.

Outline of proof:

- We consider the simply connected domain

$$\mathcal{D} = \{\lambda \in \mathbb{C} : -\infty < \Re(\lambda) < \infty,\ 0 < \Im(\lambda) < 1\}$$

and E is a closed set lying on a single component of $\partial\mathcal{D}$.

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- We consider the simply connected domain

$$\mathcal{D} = \{\lambda \in \mathbb{C} : -\infty < \Re(\lambda) < \infty,\ 0 < \Im(\lambda) < 1\}$$

and E is a closed set lying on a single component of $\partial\mathcal{D}$.

- So we get that the harmonic measure $\omega_{\mathcal{D}}(\cdot, z)$ satisfies

$$\omega_{\mathcal{D}}(E, \lambda) \longrightarrow 1 \quad \text{as } \lambda \not\measuredangle\!\!\to \lambda_0, \quad \text{for almost every } \lambda_0 \in E.$$

Since E is a set of positive Lebesgue measure, there certainly exists $a \in E,\ a \neq 0$ such that

$$\omega_{\mathcal{D}}(E, a + i\tau) \longrightarrow 1 \quad \text{as } \tau \to 0+\,. \tag{8}$$

- Since $a > 0$ it easily follows from the fact $f \in L^1_\alpha(\mathbb{R}_+)$ that $f \in L^1([a, \infty))$ which implies that the function F on the upper half plane $\mathbb{H}$ defined by

$$F(z) = \int_a^\infty e^{i\lambda z} f(\lambda)\, d\lambda, \qquad \text{for } z \in \mathbb{H}, \tag{9}$$

is analytic and bounded on $\mathbb{H}$ and continuous upto $\overline{\mathbb{H}}$. It is easy to see that if $F(z) = 0$ for $z \in \mathbb{H}$, then $f = 0$ almost everywhere on $[a, \infty)$.

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Lemma

Let F be analytic on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ and bounded in the closed half planes $\{z \in \mathbb{C} : \Im(z) \geq h\}$, for each $h > 0$. If F satisfies

$$\int_0^\infty \frac{\log(|F(x+i)|)}{1+x^2}\, dx = -\infty,$$

then F is identically zero on $\mathbb{H}$.

- $f = f_l + \rho_l$, almost everywhere, where

$$f_l(\lambda) = \int_0^l \mathcal{H}_\alpha(f)(t) j_\alpha(t\lambda)\, d\gamma_\alpha(t),$$

$$\text{and } \rho_l(\lambda) = \int_l^\infty \mathcal{H}_\alpha(f)(t) j_\alpha(t\lambda)\, d\gamma_\alpha(t), \quad \text{for } \lambda \in [0, \infty).$$

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$$F(x+i) = \int_a^\infty e^{i(x+i)\lambda} f_l(\lambda)\, d\lambda + \int_a^\infty e^{i(x+i)\lambda} \rho_l(\lambda)\, d\lambda, \text{for any } x \geq 0.$$

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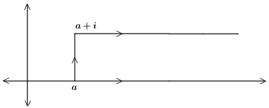
- f_l is entire function. So by Cauchy's theorem,

$$\begin{aligned} \int_a^\infty e^{i(x+i)\lambda} f_l(\lambda)\, d\lambda &= i\int_0^1 e^{i(x+i)(a+i\tau)} f_l(a+i\tau)\, d\tau \\ &+ \int_a^\infty e^{i(x+i)(\sigma+i)} f_l(\sigma+i)\, d\sigma. \end{aligned}$$



- $$|F(x+i)| \leq 3e^{-\theta M_*(x)},$$

  where $\theta = \min\limits_{0 \leq \tau \leq 1} \{\tau + \omega_{\mathcal{D}}(E, a+i\tau)\} > 0$ and

  $$M_*(x) = \min\left\{x, \log\left(\frac{1}{\int_x^\infty |\mathcal{H}_\alpha(f)(t)|\, d\gamma_\alpha(t)}\right)\right\}.$$

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- Hence $f = 0$ almost everywhere on $[a, \infty)$.
- $$\mathcal{H}_\alpha(f)(t) = \int_0^a f(\lambda) j_\alpha(\lambda t)\, d\gamma_\alpha(\lambda).$$

- $\mathcal{H}_\alpha(f)$ can be extended holomorphically to $\mathbb{C}$ satisfying
$$|\mathcal{H}_\alpha(f)(z)| \leq \int_0^a |f(\lambda)| e^{\lambda|\Im(z)|} d\gamma_\alpha(\lambda) \leq C e^{a|\Im(z)|}, \quad \text{for all } z \in \mathbb{C}.$$

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Lemma (Bhowmik, 2020)

Suppose f is a holomorphic function on $\mathbb{H}$ which extends continuously to $\overline{\mathbb{H}}$. Let ψ be a non negative even function on $\mathbb{R}$ such that for positive constants τ and C

$$|f(z)| \leq C e^{\tau|\Im(z)|}, \quad \text{for all } z \in \overline{\mathbb{H}}$$

and

$$\int_{\mathbb{R}} \frac{|f(x)| e^{\psi(x)}}{1+x^2} \, dx < \infty.$$

If

$$\int_{\mathbb{R}} \frac{\psi(x)}{1+x^2} \, dx = \infty,$$

then f vanishes identically on $\overline{\mathbb{H}}$.

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Theorem

Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(x) \to \infty$ as $x \to \infty$ and consider $I = \int_0^\infty \frac{\psi(x)}{1+x^2}\, dx$.

(a) *Let $I = \infty$ and $f \in L^1(\mathbb{R}^n)$ be a radial function satisfying*

$$\int_{\mathbb{R}^n} |\widehat{f}(x)| e^{\psi(|x|)}\, dx < \infty. \tag{10}$$

If f vanishes on a set of positive Lebesgue measure in $\mathbb{R}^n$, then f is zero almost everywhere on $\mathbb{R}^n$.

(b) *If I is finite then there exists a non-trivial radial function f on $\mathbb{R}^n$ satisfying (10) which vanishes on a set of positive Lebesgue measure in $\mathbb{R}^n$.*

Some more analogues for Fourier transform on $\mathbb{R}^n$

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Theorem

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(a) *If $I = \infty$, $f \in \mathcal{S}(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} |\widehat{f}(x)| e^{\psi(|x|)}\, dx < \infty \tag{11}$$

and f vanishes on an annular type set of positive Lebesgue measure in $\mathbb{R}^n$, then f is identically zero.

(b) *If $I < \infty$, then $\exists$ a non-trivial $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying (11) which vanishes on an annular type set of positive Lebesgue measure.*

$E' \subset \mathbb{R}^n$ is called an annular type set if $x \in E'$ implies that $|x|\omega \in E'$, for all $\omega \in S^{n-1}$.

Analogue for spectral projections associated to Laplacian

- For suitable function f, spectral projections associated to the Euclidean Laplacian given by

$$f * \phi_\lambda(x) = \int_{S^{n-1}} \widehat{f}(\lambda\omega) e^{i\lambda x\cdot\omega} d\sigma(\omega),$$

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Theorem

Let ψ and I defined as before.

(a) *Let $I = \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying*

$$|f * \phi_\lambda(x)| \leq Ce^{-\psi(\lambda)}, \quad \text{for } \lambda \geq 0, \; x \in \mathbb{R}^n. \tag{12}$$

If f vanishes on an annular type set of positive Lebesgue measure in $\mathbb{R}^n$, then f is identically zero.

(b) *$I < \infty$, then $\exists$ a non-trivial function $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying (12) which vanishes on a set of positive Lebesgue measure in $\mathbb{R}^n$.*

Analogue for Jacobi transform

- We consider the Jacobi operator

$$\mathcal{L} = \frac{d^2}{dt^2} + \frac{A'(t)}{A(t)}\frac{d}{dt},$$

where for $\alpha \geq \beta \geq -\frac{1}{2}$,

$$A(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}, \quad \text{for } t \in [0, \infty).$$

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- For each $\lambda \in \mathbb{C}$, $2\rho = \alpha + \beta + 1$, we define ϕ_λ as the unique solution of

$$\mathcal{L}f + (\lambda^2 + \rho^2) f = 0, \quad \text{with } f(0) = 1, f'(0) = 0.$$

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- For a function $f \in L^1(\mathbb{R}_+, A(t)dt)$, we define the Jacobi transform of f by

$$\widehat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda(t)A(t)\,dt, \quad \text{for } \lambda \in \mathbb{R}_+.$$

For $f \in L^1(\mathbb{R}_+, A(t)dt)$ and $\widehat{f} \in L^1(\mathbb{R}_+, |c(\lambda)|^{-2}d\lambda)$ we have the following inversion formula

$$f(t) = \int_0^\infty \widehat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}\, d\lambda, \quad \text{for a.e. } t \in \mathbb{R}_+, \tag{13}$$

where $c(\lambda)$ is the Harish-Chandra c-function.

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Theorem

Let l and ψ be defined as before.

(a) *Let $l = \infty$ and $f \in L^1(\mathbb{R}_+, A(t)dt)$ satisfying*

$$\int_0^\infty |\widehat{f}(\lambda)|e^{\psi(\lambda)}|c(\lambda)|^{-2}\, d\lambda < \infty. \tag{14}$$

If f vanishes on a set $E \subseteq [0, \infty)$ of positive Lebesgue measure, then $f = 0$ almost everywhere on $[0, \infty)$.

(b) *If l is finite, then there exists a function f on $\mathbb{R}_+$ satisfying (14) which vanishes on a set of positive Lebesgue measure.*

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Thank You !!!