Pseudo-differential operators, Wigner transform and Weyl transform on the Similitude group, $\mathbb{SIM}(2)$

By
Santosh Kumar Nayak
(Joint work with Prof. Aparajita Dasgupta)

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Department of Mathematics
Indian Institute of Technology Delhi

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What is Pseudo-Differential Operator?

- Recall a partial differential operator $P(x, D)$ on $\mathbb{R}^n$ is given by

$$P(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha,$$

where the coefficients $a_\alpha(x)$ are smooth and bounded functions defined on $\mathbb{R}^n$.

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where the coefficients $a_\alpha(x)$ are smooth and bounded functions defined on $\mathbb{R}^n$.

- Let us take any function ϕ in $\mathcal{S}(\mathbb{R}^n)$ and now

$$\begin{aligned}(P(x, D)\phi)(x) &= \sum_{|\alpha| \leqslant m} a_\alpha(x)(D^\alpha \phi)(x) \\ &= \sum_{|\alpha| \leqslant m} a_\alpha(x)\left(\widehat{D^\alpha \phi}\right)^\vee (x) \\ &= \sum_{|\alpha| \leqslant m} a_\alpha(x)(\xi^\alpha \widehat{\phi})^\vee (x).\end{aligned}$$

What is Pseudo-Differential Operator?

- $$\begin{aligned}(P(x,D)\phi)(x) &= \sum_{|\alpha|\leqslant m} a_\alpha(x)(2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(ix\cdot\xi)\xi^\alpha\hat{\phi}(\xi)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\Big(\sum_{|\alpha|\leqslant m} a_\alpha(x)\xi^\alpha\Big)\exp(ix\cdot\xi)\hat{\phi}(\xi)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} P(x,\xi)\exp(ix\cdot\xi)\hat{\phi}(\xi)d\xi.\end{aligned}$$

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Example 1

Let P be the symbol defined by

$$P(x,\xi) = c\exp(-|\xi|^2),$$

for all ξ in $\mathbb{R}^n$ and for some $c > 0$. Then $P(x,D)$ is an pseudo-differential operator, but not partial differential operator.

Motivation

- The same idea is used to define the ΨDOs on $\mathbb{T}^n, \mathbb{Z}^n$, Heisenberg group, graded Lie group, compact group, for unimodular groups, affine group, Poincaré unit disk.
- A basic result in theory of ΨDOs on $\mathbb{R}^n$ is that, if the symbol σ in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then the corresponding operator T_σ can be extended to a bounded linear operator $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and moreover it is a Hilbert-Schmidt operator. The main result is to establish the conditions on the symbol so that ΨDO is a Hilbert-Schmidt operator on $\mathbb{SIM}(2)$.

Similitude Group $\mathbb{SIM}(2)$

- $\mathbb{SIM}(2) = \{(b, a, \theta) : b \in \mathbb{R}^2, a > 0, 0 \leqslant \theta < 2\pi\}$

$$\begin{aligned} (b, a, \theta) * (b', a', \theta') &= (b + aR_\theta b', aa', \theta + \theta') \\ e &= (0, 1, 0) \\ (b, a, \theta)^{-1} &= (-a^{-1} R_{-\theta} b, a^{-1}, -\theta). \end{aligned}$$

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- Its left and right Haar measure given by

$$d\mu_L(b, a, \theta) = \frac{dbdad\theta}{a^3}, \quad d\mu_R(b, a, \theta) = \frac{dbdad\theta}{a},$$

respectively, are different. The modular function on $\mathbb{SIM}(2)$, denoted by Δ is given by

$$\Delta(b, a, \theta) = \frac{1}{a^2}, \quad (b, a, \theta) \in \mathbb{SIM}(2).$$

Unitary representation of $\mathbb{SIM}(2)$

- Let $\pi : \mathbb{SIM}(2) \to U(L^2(\mathbb{R}^2))$ be the mapping of $\mathbb{SIM}(2)$ into the group $U(L^2(\mathbb{R}^2))$, of all unitary operators on $L^2(\mathbb{R}^2)$ is given by

$$\big(\pi(b,a,\theta)\phi\big)(x) = ae^{-ib\cdot x}\phi(aR_{-\theta}x), \quad x \in \mathbb{R}^2,$$

for all (b,a,θ) in $\mathbb{SIM}(2)$ and all ϕ in $L^2(\mathbb{R}^2)$.

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for all (b, a, θ) in $\mathbb{SIM}(2)$ and all ϕ in $L^2(\mathbb{R}^2)$.

Theorem 2

π is the only infinite dimensional, irreducible and unitary representation of $\mathbb{SIM}(2)$.

Fourier transform on $\mathbb{SIM}(2)$

- Let $\phi \in L^2(\mathbb{R}^2)$. Define the function $D\phi$ on $\mathbb{R}^2$ by,

$$(D\phi)(x) = \|x\|\phi(x), \tag{2.1}$$

and it can be verified that

$$\Delta(b,a,\theta)^{1/2} D\pi(b,a,\theta) = \pi(b,a,\theta)D, \ (b,a,\theta) \in \mathbb{SIM}(2).$$

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- Let $f \in L^1(\mathbb{SIM}(2)) \cap L^2(\mathbb{SIM}(2))$ and define the Fourier transform $\hat{f}$ of f on $\widehat{\mathbb{SIM}(2)} = \{\pi\}$ by

$$(\hat{f}(\pi)\phi)(x) = \int_{\mathbb{SIM}(2)} f(b,a,\theta)(\pi(b,a,\theta)D\phi)(x)\frac{dbdad\theta}{a^3},$$

for all $\phi \in L^2(\mathbb{R}^2)$.

Plancherel Formula

Theorem 3

Let $f \in L^1(\mathbb{SIM}(2)) \cap L^2(\mathbb{SIM}(2))$. Then $\widehat{f}(\pi)$ is a Hilbert Schmidt operator with kernel given by

$$K^f(x,y) = \begin{cases} (\mathcal{F}_1 f)\left(x, \sqrt{\frac{y\cdot y}{x\cdot x}}, \cos^{-1}\left(\frac{x\cdot y}{\|x\|\|y\|}\right)\right)\frac{\|x\|}{\|y\|^2}, & x \neq 0, y \neq 0 \\ 0, & \textit{otherwise}. \end{cases} \tag{2.2}$$

Moreover $\|\widehat{f}(\pi)\|^2_{\mathcal{HS}} = \|f\|^2_{L^2(\mathbb{SIM}(2))}$

Inversion Formula

Lemma 4

$D\hat{f}(\pi)\pi(b,a,\theta)^*$ is an integral operator with kernel

$$S^{f,\pi}(x,w) = a\|x\|K^f(x, R_{-\theta}aw)e^{ib\cdot w}, \quad x,w \in \mathbb{R}^2,$$

where $K^f(x,y)$ is defined in (2.2), and $f \in L^2(\mathbb{SIM}(2))$ and $z = (b,a,\theta)$ in $\mathbb{SIM}(2)$.

Inversion Formula

Lemma 4

$D\hat{f}(\pi)\pi(b,a,\theta)^*$ is an integral operator with kernel

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where $K^f(x,y)$ is defined in (2.2), and $f \in L^2(\mathbb{SIM}(2))$ and $z = (b,a,\theta)$ in $\mathbb{SIM}(2)$.

Next we obtained the Fourier inversion formula.

Theorem 5

Let $f \in L^2(\mathbb{SIM}(2))$. Then

$$f(b,a,\theta) = \Delta(b,a,\theta)^{-\frac{1}{2}}\, Tr(D\hat{f}(\pi)\pi(b,a,\theta)^*),$$

for all $(b,a,\theta) \in \mathbb{SIM}(2)$.

ΨDO on $\mathbb{SIM}(2)$

- Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2))$ be an operator valued symbol such that

$$\sigma((b,a,\theta),\pi) \in B(L^2(\mathbb{R}^2)), \quad (b,a,\theta) \in \mathbb{SIM}(2),$$

where $B(L^2(\mathbb{R}^2))$ is the C^*-algebra of all bounded linear operators from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$.

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  where $B(L^2(\mathbb{R}^2))$ is the C^*-algebra of all bounded linear operators from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$.

- We define the pseudo-differential operator T_σ corresponding to the operator valued symbol symbol, σ as;

$$(T_\sigma f)(b, a, \theta) = \Delta(b, a, \theta)^{-\frac{1}{2}} \mathrm{Tr}\left(D\sigma(b, a, \theta, \pi)\hat{f}(\pi)\pi(b, a, \theta)^*\right),$$

  for all $f \in L^2(\mathbb{SIM}(2))$ and $(b, a, \theta) \in \mathbb{SIM}(2)$.

$L^2 - L^2$ Boundedness of ΨDO

Theorem 6

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2))$ *be an operator-valued symbol such that for all* $(b, a, \theta) \in \mathbb{SIM}(2)$, $D\sigma(b, a, \theta)$ *is an integral operator, from* $L^2(\mathbb{R}^2)$ *into* $L^2(\mathbb{R}^2)$ *with kernel* $S_{\sigma(b,a,\theta,\pi)}$ *in* $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$*. Furthermore, suppose the function* G *on* $\mathbb{SIM}(2)$ *defined by*

$$G(b, a, \theta) = \Delta(b, a, \theta)^{\frac{-1}{2}} \|S_{\sigma(b,a,\theta,\pi)}\|,$$

is in $L^2(\mathbb{SIM}(2))$*. Then the pseudo-differential operator corresponding to given symbol* σ*,*

$$T_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$$

is a bounded operator and

$$\|T_\sigma\|_{B(L^2(\mathbb{SIM}(2)))} \leqslant \|G\|_{L^2(\mathbb{SIM}(2))}.$$

$L^2 - L^p$ Boundedness of ΨDO

Theorem 7

Let $2 \leqslant p \leqslant \infty$ and q be conjugate index of p, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $\sigma : \mathbb{SIM}(2) \times \{\pi\} \to B(L^2(\mathbb{R}^2))$ be an operator valued symbol such that for all (b, a, θ) in $\mathbb{SIM}(2)$

$$D\sigma(b, a, \theta, \pi) \in S_q(\mathbb{R}^2),$$

where D is the Duflo-Moore operator defined in 2.1 and the function G on $\mathbb{SIM}(2)$ defined by

$$G(b, a, \theta) = \Delta^{\frac{1}{2}} \|D\sigma(b, a, \theta, \pi)\|_{S_q} \in L^p(\mathbb{SIM}(2)).$$

Then the pseudo-differential operator $T_\sigma : L^2(\mathbb{SIM}(2)) \to L^p(\mathbb{SIM}(2))$ is a bounded linear operator.

Theorem 8

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2))$ *be a symbol such that satisfies assumption of the Theorem 6. Furthermore, suppose that*

$$\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \ni (b, a, \theta, \pi) \mapsto (D\sigma(b, a, \theta, \pi)\pi(b, a, \theta)^* \in S_2 \tag{2.3}$$

is weakly continuous. Then $T_\sigma f = 0$ *for all* f *in* $L^2(\mathbb{SIM}(2))$ *if and only if* $\sigma(b, a, \theta, \pi) = 0$ *for almost all* (b, a, θ, π) *in* $\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)}$.

Corollary 8.1

Let $f \in L^2(\mathbb{SIM}(2))$. Then

$$\widehat{f}(\pi) = W_{\sigma_f},$$

where

$$\sigma_f = (2\pi)\mathcal{F}_2 TK^f,$$

and $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the unitary twisting operator defined by

$$(Tg)(s, t) = g\left(s + \frac{t}{2}, s - \frac{t}{2}\right), \quad (s, t) \in \mathbb{R} \times \mathbb{R}.$$

Theorem 9

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to S_2$ *be an operator valued symbol such that the hypothesis of Theorem 8 are satisfied. Then the corresponding pseudo-differential operator* $T_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$ *is a Hilbert-Schmidt operator if and only if*

$$D\sigma(b, a, \theta, \pi) = \pi(b, a, \theta) W_{\tau_{\alpha(b,a,\theta)}}, (b, a, \theta) \in \mathbb{SIM}(2),$$

where $\tau_{\alpha(b,a,\theta)}(x, y) = \mathcal{F}_2^{-1} TK_{\alpha(b,a,\theta)}(x, y)$, *and*

$$K_{\alpha(b,a,\theta)}(x, y) = \begin{cases} \mathcal{F}_1^{-1} \frac{\alpha(b,a,\theta)}{a} \left(x, \sqrt{\frac{y \cdot y}{x \cdot x}}, \cos^{-1}\left(\frac{x.y}{\|x\|\|y\|}\right)\right), & \text{if } x \neq 0, y \neq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Also, $\alpha : \mathbb{SIM}(2) \to L^2(\mathbb{SIM}(2))$ *is a weakly continuous mapping for which*

$$\int_{\mathbb{SIM}(2)} \|\alpha(b, a, \theta)\|^2_{L^2(\mathbb{SIM}(2))} \frac{dbdad\theta}{a^3} < \infty.$$

Proof:

- We first show the sufficiency part. Let $f \in S(\mathbb{SIM}(2))$, Schwartz space on $\mathbb{SIM}(2)$.

$$\begin{aligned} (T_\sigma f)(b,a,\theta) &= \Delta(b,a,\theta)^{-\frac{1}{2}} \mathrm{Tr}\left(D\sigma(b,a,\theta,\pi)\hat{f}(\pi)\pi(b,a,\theta)^*\right) \\ &= \Delta(b,a,\theta)^{-\frac{1}{2}} \mathrm{Tr}\left(\pi(b,a,\theta)^* D\sigma(b,a,\theta,\pi)\hat{f}(\pi)\right) \\ &= \Delta(b,a,\theta)^{-\frac{1}{2}} \mathrm{Tr}\left(W_{\tau_{\alpha(b,a,\theta)}} W_{\sigma_f}\right). \end{aligned}$$

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-

$$\begin{aligned}&\mathrm{Tr}\left(W_{\tau_{\alpha(b,a,\theta)}}W_{\sigma_f}\right) \qquad (2.4)\\ &= \int_{\mathbb{R}^4}\tau_{\alpha(b,a,\theta)}(x,y)\sigma_f(x,y)\,dxdy\\ &= \int_{\mathbb{R}^4}\mathcal{F}_2^{-1}TK_{\alpha(b,a,\theta)}(x,y)\mathcal{F}_2TK^f(x,y)\,dxdy\end{aligned}$$

- $$= \frac{1}{a} \int_{\mathbb{SIM}(2)} \alpha(b, a, \theta)(x, a', \theta') f(x, a', \theta') \frac{dx da' d\theta'}{a'^3}, \qquad (2.5)$$

-

$$= \frac{1}{a} \int_{\mathbb{SIM}(2)} \alpha(b,a,\theta)(x,a',\theta') f(x,a',\theta') \frac{dx\,da'\,d\theta'}{a'^3}, \qquad (2.5)$$

- So the kernel of T_σ is the function k on $\mathbb{SIM}(2) \times \mathbb{SIM}(2)$ given by

$$k(b,a,\theta,b',a',\theta') = \alpha(b,a,\theta)(b',a',\theta'), \quad (b,a,\theta),(b',a',\theta') \in \mathbb{SIM}(2). \qquad (2.6)$$

- $$= \frac{1}{a} \int_{\mathbb{SIM}(2)} \alpha(b,a,\theta)(x,a',\theta') f(x,a',\theta') \frac{dx da' d\theta'}{a'^3}, \qquad (2.5)$$

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- By Fubini's theorem and Plancheral theorem,

$$\int_{\mathbb{SIM}(2)} \int_{\mathbb{SIM}(2)} |k(b,a,\theta,b',a',\theta')|^2 \frac{db da d\theta}{a^3} \frac{db' da' d\theta'}{a'^3}$$

$$= \int_{\mathbb{SIM}(2)} \|\alpha(b,a,\theta)\|^2_{L^2(\mathbb{SIM}(2))} \frac{db da d\theta}{a^3} < \infty.$$

Conversely, suppose that $T_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$ is a Hilbert-Schmidt operator. Then there exists a function k in $L^2(\mathbb{SIM}(2) \times \mathbb{SIM}(2))$ such that for all f in $L^2(\mathbb{SIM}(2))$,

$$(T_\sigma f)(b, a, \theta) = \int_{\mathbb{SIM}(2)} k(b, a, \theta, b', a', \theta') f(b', a', \theta') \frac{db' da' d\theta'}{a'^3}.$$

Let $\alpha : \mathbb{SIM}(2) \to L^2(\mathbb{SIM}(2))$ is a mapping defined by

$$\alpha(b, a, \theta)(b', a', \theta') = k(b, a, \theta, b', a', \theta').$$

Reversing the sufficiency part, we get

$$(T_\sigma f)(b, a, \theta) = a \left(\mathrm{Tr} \left(W_{\tau_{\alpha(b,a,\theta)}} W_{\sigma_f} \right) \right),$$

which gives

$$D\sigma(b, a, \theta, \pi) = \pi(b, a, \theta) W_{\tau_{\alpha(b,a,\theta)}},$$

for $(b, a, \theta) \in \mathbb{SIM}(2)$.

Theorem 10

Let $\alpha \in L^2(\mathbb{SIM}(2) \times \mathbb{SIM}(2))$ *be such that*

$$\int_{\mathbb{SIM}(2)} |\alpha(b,a,\theta,b,a,\theta)| \frac{dbdad\theta}{a^3} < \infty.$$

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2))$ *be a symbol in preceeding theorem. Then* $T_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$ *is a trace class operator and*

$$Tr(T_\sigma) = \int_{\mathbb{SIM}(2)} \alpha(b,a,\theta,b,a,\theta) \frac{dbdad\theta}{a^3}.$$

Proof.

The proof follows from the formula (2.6) on the kernel of the pseudo-differential operator in the proof of the preceding theorem. □

Theorem 11

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to S_2$ *be a symbol satisfying the hypothesis of Theorem 8. Then pseudo-differential operator* $T_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$ *is a trace class operator if and only if*

$$D\sigma(b, a, \theta, \pi) = \pi(b, a, \theta) W_{\tau_{\alpha(b,a,\theta)}},$$

where $\alpha : \mathbb{SIM}(2) \to L^2(\mathbb{SIM}(2))$ *is a mapping such that the conditions of Theorem 9 (Hilbert-Schmidt Operator) are satisfied and*

$$\alpha(b, a, \theta)(b', a', \theta') = \int_{\mathbb{SIM}(2)} \alpha_1(b, a, \theta)(b'', a'', \theta'') \alpha_2(b'', a'', \theta'')(b', a', \theta') \frac{db'' da'' d\theta''}{a''^3}$$

for all $(b, a, \theta), (b', a', \theta') \in \mathbb{SIM}(2)$*. Here* $\alpha_i : \mathbb{SIM}(2) \to L^2(\mathbb{SIM}(2))$*,* $i = 1, 2$*, are such that*

$$\int_{\mathbb{SIM}(2)} \|\alpha_i(b, a, \theta)\|^2_{L^2(\mathbb{SIM}(2))} \frac{db da d\theta}{a^3} < \infty, i = 1, 2.$$

Weyl transform

The classical Weyl transform was first introduced by Herman Weyl, which was arising in quantum mechanics. This theory has been vastly used in PDE and physics. Weyl transforms are look like pseudo-differential operators on $\mathbb{R}^n$, or you can say a type of ΨDOs. But these are not same, because former one is selfadjoint whereas other one is not. Weyl transform were investigated in Heisenberg group, affine group, Poincaré unit disk, locally compact abelian group. We investigated the Weyl transform on polar affine group.

Weyl Transform on $\mathbb{R}^n$

Let σ be in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ corresponding to σ is defined by

$$\langle W_\sigma f, g \rangle_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \iint\limits_{\mathbb{R}^{2n}} \sigma(x,\xi) W(f,g)(x,\xi) dx d\xi,$$

for all $f, g \in L^2(\mathbb{R}^n)$, where $W(f,g)$ is the Wigner transform of f and g, is defined by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x + p/2\right) \overline{g\left(x - p/2\right)} dp, \quad x, \xi \in \mathbb{R}^n.$$

The Fourier-Wigner transform $V(f,g)$ of f and g, are in $L^2(\mathbb{R}^n)$, is defined by

$$V(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y + p/2\right) \overline{g\left(y - p/2\right)} dy, \quad q, p \in \mathbb{R}^n.$$

For all (b, a, θ) in $\mathbb{SIM}(2)$, there exists a unique element $(\tilde{b}, \tilde{a}, \tilde{\theta}) \in \mathbb{SIM}(2)$ such that

$$(\tilde{b}, \tilde{a}, \tilde{\theta}) * (\tilde{b}, \tilde{a}, \tilde{\theta}) = (b, a, \theta).$$

Then

$$\tilde{a} = \sqrt{a}, \tilde{\theta} = \frac{\theta}{2}, \tilde{b} = \frac{1}{1 + a + 2\sqrt{a}\cos\frac{\theta}{2}}(b + \sqrt{a}R_{-\theta/2}b).$$

Denote, $(b, a, \theta)^{1/2} = \left(\frac{1}{1+a+2\sqrt{a}\cos\frac{\theta}{2}}(b + \sqrt{a}R_{-\theta/2}b), \sqrt{a}, \frac{\theta}{2}\right)$. With this preparation, we are ready to define Fourier-Wigner transforms on $\mathbb{SIM}(2)$.

- Let $f, g \in L^2(\mathbb{SIM}(2))$. We define the Fourier-Wigner transform $V(f,g)$ of f and g on $\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2)$ as the mapping,

$$V(f,g) : \widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2) \rightarrow B(L^2(\mathbb{R}^2)),$$

by

$$(V(f,g)(\pi,\xi)\phi)(x) = \int_{\mathbb{SIM}(2)} f(\xi^{\frac{1}{2}} * w)\overline{g(z^{-1} * \xi^{\frac{1}{2}})}(\pi(w)D\phi)(x)\,d\mu_L(w),$$

for all $\phi \in L^2(\mathbb{R}^2)$ and $\xi \in \mathbb{SIM}(2)$, where the left Haar measure $d\mu_L(w) = \frac{db'\,da'\,d\theta'}{a'^3}$.

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$$V(f, g) : \widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2) \to B(L^2(\mathbb{R}^2)),$$

by

$$(V(f, g)(\pi, \xi)\phi)(x) = \int_{\mathbb{SIM}(2)} f(\xi^{\frac{1}{2}} * w)\overline{g(z^{-1} * \xi^{\frac{1}{2}})}(\pi(w)D\phi)(x)d\mu_L(w),$$

for all $\phi \in L^2(\mathbb{R}^2)$ and $\xi \in \mathbb{SIM}(2)$, where the left Haar measure $d\mu_L(w) = \frac{db'\,da'\,d\theta'}{a'^3}$.

- Let F^ξ be a function on $\mathbb{SIM}(2)$ given by

$$F^\xi(w) = f(\xi^{\frac{1}{2}} * w)\overline{g(w^{-1} * \xi^{\frac{1}{2}})},$$

then

$$V(f, g)(\pi, \xi) = (\mathcal{F}_{\mathbb{SIM}(2)} F^\xi)(\pi).$$

- Define the vector space, $L^2\left(\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2), \mathrm{HS}(L^2(\mathbb{R}^2))\right)$, over $\mathbb{C}$ by,

$$L^2\left(\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2), \mathcal{HS}(L^2(\mathbb{R}^2))\right)$$
$$= \{F | F : \widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2) \to \mathcal{HS}(L^2(\mathbb{R}^2))\},$$

where $\mathcal{HS}(L^2(\mathbb{R}^2))$ be the set of Hilbert-Schmidt operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, and the inner product is defined by

$$\langle F, G \rangle = \int_{\mathbb{SIM}(2)} \mathrm{Tr}(F(\pi, w) G(\pi, w)^*) d\mu_L(w).$$

- Define the vector space, $L^2\left(\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2), \mathrm{HS}(L^2(\mathbb{R}^2))\right)$, over $\mathbb{C}$ by,

$$\begin{aligned} & L^2\left(\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2), \mathcal{HS}(L^2(\mathbb{R}^2))\right) \\ & = \{F | F : \widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2) \to \mathcal{HS}(L^2(\mathbb{R}^2))\}, \end{aligned}$$

where $\mathcal{HS}(L^2(\mathbb{R}^2))$ be the set of Hilbert-Schmidt operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, and the inner product is defined by

$$\langle F, G\rangle = \int_{\mathbb{SIM}(2)} \mathrm{Tr}(F(\pi, w)G(\pi, w)^*) d\mu_L(w).$$

We then have the following Moyal identity.

Theorem 12

Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{SIM}(2))$. Then

$$\langle V(f_1, g_1), V(f_2, g_2)\rangle_{L^2(\widehat{\mathbb{SIM}(2)} \times \mathbb{SIM}(2), \mathcal{HS})} = \langle f_1, f_2\rangle_{L^2(\mathbb{SIM}(2))} \overline{\langle g_1, g_2\rangle_{L^2(\mathbb{SIM}(2))}}.$$

- Let $f, g \in L^2(\mathbb{SIM}(2))$. Then the Wigner transform $W(f,g)$ of f and g on $\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)}$ is a mapping,

$$W(f,g) : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2)),$$

defined by

$$W(f,g)(w,\pi) = \left(\mathcal{F}_{\mathbb{SIM}(2),2}\mathcal{F}^{-1}_{\mathbb{SIM}(2),1} V(f,g)\right)(w,\pi).$$

- Let $L^2\big(\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)}, \mathcal{HS}(L^2(\mathbb{R}^2))\big)$ be the space of measurable functions $F : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to \mathcal{HS}(L^2(\mathbb{R}^2))$ such that

$$F(w, \pi) \in \mathcal{HS}(L^2(\mathbb{R}^2)), \qquad w \in \mathbb{SIM}(2)).$$

Then for all $F, G \in L^2\big(\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)}, \mathcal{HS}(L^2(\mathbb{R}^2))\big)$, we define the inner product of F and G by

$$\langle F, G\rangle_{L^2\left(\mathbb{SIM}(2)\times\widehat{\mathbb{SIM}(2)}\right)} = \int_{\mathbb{SIM}(2)} \mathrm{Tr}(F(b,a,\theta,\pi)G(b,a,\theta,\pi)^*)d\mu_L(b,a,$$

Next we prove the following Moyal's identity for the Wigner transform.

Theorem 13

Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{SIM}(2))$. Then

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle_{L^2\left(\mathbb{SIM}(2)\times\widehat{\mathbb{SIM}(2)},\mathcal{HS}\right)} = \langle f_1, f_2\rangle_{L^2(\mathbb{SIM}(2))}\langle g_1, g_2\rangle_{L^2(\mathbb{SIM}(2)}$$

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \rightarrow B\left(L^2(\mathbb{R}^2)\right)$ be an operator-valued symbol. Then the Weyl transform corresponding to the symbol σ is defined by

$$\langle W_\sigma f, g \rangle_{L^2(\mathbb{SIM}(2))} = \int_{\mathbb{SIM}(2)} \operatorname{Tr}\left(D\sigma(b, a, \theta) W(f, g)(b, a, \theta, \pi)\right) d\mu_L(b, a, \theta),$$

for all functions f and g in $L^2(\mathbb{SIM}(2))$, where $W(f, g)$ is the Wigner transform of f and g defined earlier.

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B\left(L^2(\mathbb{R}^2)\right)$ be an operator-valued symbol. Then the Weyl transform corresponding to the symbol σ is defined by

$$\langle W_\sigma f, g\rangle_{L^2(\mathbb{SIM}(2))} = \int\limits_{\mathbb{SIM}(2)} \mathrm{Tr}\left(D\sigma(b,a,\theta) W(f,g)(b,a,\theta,\pi)\right) d\mu_L(b,a,\theta),$$

for all functions f and g in $L^2(\mathbb{SIM}(2))$, where $W(f,g)$ is the Wigner transform of f and g defined earlier.

Theorem 14

Let $\sigma : \mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)} \to B(L^2(\mathbb{R}^2))$ be an operator valued symbol such that $D\sigma \in L^2(\mathbb{SIM}(2) \times \widehat{\mathbb{SIM}(2)}, HS)$, then $W_\sigma : L^2(\mathbb{SIM}(2)) \to L^2(\mathbb{SIM}(2))$ is a bounded linear operator.

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Thank You