Tensor products of local operator systems

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Subspaces of B(H)

- C^*-**algebra**[1] is a closed $*$- subalgebra of $B(H)$
- **Operator space**[2] is a closed subspace of $B(H)$
- **Operator system**[3] is a unital $*$- closed subspace of $B(H)$

[1] Israel Gelfand and Mark Naimark in 1943
[2] Ruan in 1988
[3] Choi and Effros in 1977

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Subalgebras of $B(H)$ which is closed under the operator norm and under adjoints.

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For every C^*-algebra A there exists a Hilbert space H such that A is isometrically $*$-isomorphic to some C^*-subalgebra of $B(H)$.

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An abstract operator space is a normed space V with a sequence of norm $\|.\|_n : M_n(V) \to [0, \infty)$ satisfying:

1. $\|v \oplus w\|_{n+m} \leq \max \{\|v\|_n, \|w\|_m\}$
2. $\|\alpha v \beta\|_m \leq \|\alpha\| \|v\|_n \|\beta\|$

where $v \in M_n(V), w \in M_m(V), \alpha \in M_{m,n}, \beta \in M_{n,m}$.

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Remark (Morphisms)

A linear map $\phi : V \rightarrow W$ is said to be completely bounded if

$$\|\phi\|_{cb} := Sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty$$

where $\phi_n : M_n(V) \rightarrow M_n(W)$ is defined as $\phi_n((x_{ij})) = (\phi(x_{ij}))$.

Operator system

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Definition (Abstract)

An abstract operator system is a triple $(V, \{C_n\}_{n=1}^{\infty}, e)$ where V is a complex $*$-vector space and $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V and e is Archimedean matrix order unit.

Definition

An ordered $*$-vector space is a pair (V, V^+) consisting of a $*$-vector space and a subset $V^+ \subseteq V_h$ satisfying the following two properties:

1. V^+ is a cone in V_h
2. $V^+ \cap -V^+ = \{0\}$

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Definition

$*$-matrix ordering: $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V if

1. C_n is a cone in $M_n(V)_h$ for all $n \in \mathbb{N}$
2. $C_n \cap -C_n = \{0\}$ for all $n \in \mathbb{N}$
3. $X \in M_{n,m}$ for each $n, m \in \mathbb{N}$ we have $X^* C_n X \subseteq C_m$.

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- Order unit : $e \in V_h$ if for all $v \in V_h$ there exists $r > 0$ such that $re \geq v$.
- Archimedean order unit: $v \in V$ and $re + v \geq 0$ for all $r > 0$ implies $v \in V^+$.
- Archimedean matrix order unit:
  $e_n = \mathrm{diag}(e, e, \ldots, e)$

Theorem (Choi-Effros)

Every concrete operator system V is an abstract operator system. Conversely, if $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an abstract operator system, then there exists a Hilbert space H, a concrete operator system $S \subseteq B(H)$, and a complete order isomorphism $\phi : V \to S$ with $\phi(e) = I$.

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Remark (Morphisms)

- $\phi : V \to W$ *is positive if* $\phi(V^+) \subseteq W^+$.
- *If* $\phi_n : M_n(V) \to M_n(W)$ *is positive for all* n *then* ϕ *is said to be completely positive.*
- ϕ *is called complete order isomorphism if* ϕ *is invertible and both* ϕ *and* ϕ^{-1} *are completely positive.*

Projective limit

Definition

Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of objects of a category $\mathcal{C}$, where $(\Lambda, \leq)$ is a directed set. Also we have family of morphism $\{f_{\alpha\beta} : A_\beta \to A_\alpha : \alpha \leq \beta\}$ such that

1. $f_{\alpha\alpha}$ is the identity on A_α,
2. $f_{\alpha\beta} = f_{\alpha\gamma} \circ f_{\gamma\beta}$, for all $\alpha \leq \gamma \leq \beta$

The projective limit of $(\{A_{\alpha\in\Lambda}\}, \{f_{\alpha\gamma} : \alpha \leq \gamma\})$ is denoted by $A = \lim\limits_{\leftarrow} A_\alpha$ and also, as a set, A equals to:

$$A = \{(x_\alpha) \in \Pi_{\alpha\in\Lambda} A_\alpha : f_{\alpha\gamma}(x_\gamma) = x_\alpha \quad \forall \alpha \leq \gamma\}$$

Pro C*-algebras

- The categories of C^*-algebras, Operator spaces, Operator systems are not closed under projective limits.
- Inoue in 1972 introduced locally C^*-algebras abstractly as complete locally $*$-algebra with C^* condition where topology is defined by a family of C^*-semi norms.
- Arveson called these algebras as Pro C^*-algebra which can be represented as projective limit of C^*-algebras.

Local operator spaces and local operator systems

- Local operator spaces are projective limits of operator spaces, Webster and Effros did a systematic study on local operator spaces.
- Dosiev gave a representation theorem for local operator spaces that extends Ruan's representation theorem for operator spaces.
- Dosiev also introduced the locally convex version of operator system called as concrete local operator system.

For a fixed Hilbert space H, an upward filtered family of closed subspaces $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ such that their union $\mathcal{D}$ is a dense subspace in H with $p = \{P_\alpha\}_{\alpha\in\Lambda}$ family of projections in $B(H)$ onto the subspaces $H_\alpha, \alpha \in \Lambda$. The algebra $C_{\mathcal{E}}(\mathcal{D})$ of all non-commutative continuous functions on a quantized domain E is given by

$$C_{\mathcal{E}}(\mathcal{D}) = \{T \in L(\mathcal{D}) : TP_\alpha = P_\alpha TP_\alpha \in B(H), \alpha \in \Lambda\},$$

where $L(\mathcal{D})$ is the associative algebra of all linear transformations on $\mathcal{D}$. Thus each $T \in C_{\mathcal{E}}(\mathcal{D})$ is an unbounded operator on H with domain $\mathcal{D}$ such that $T(H_\alpha) \subseteq H_\alpha$ and $T|_{H_\alpha} \in B(H_\alpha)$, and $C_{\mathcal{E}}(\mathcal{D})$ is a subalgebra in $L(\mathcal{D})$. The set

$$C^*_{\mathcal{E}}(\mathcal{D}) = \{T \in C_{\mathcal{E}}(\mathcal{D}) \ : \ P_\alpha T \subseteq TP_\alpha, \alpha \in \Lambda\}$$

of all non-commutative continuous functions on E is a unital $*$-subalgebra of $C_{\mathcal{E}}(\mathcal{D})$, with the involution $T^* = T^\bigstar|_D \in C^*_{\mathcal{E}}(\mathcal{D})$ for all $T \in C^*_{\mathcal{E}}(\mathcal{D})$ where $T^\bigstar$ is unbounded dual of T such that $\mathcal{D} \subseteq \mathrm{dom}(T^\bigstar)$ and $T^\bigstar(\mathcal{D}) \subseteq \mathcal{D}$.

Concrete structures

1. Pro C^*-algebra is a $*$-closed subalgebra of $C^*_{\mathcal{E}}(\mathcal{D})$.
2. Local operator space is a subspace of $C_{\mathcal{E}}(\mathcal{D})$.
3. Local operator system is a unital self adjoint subspace of $C^*_{\mathcal{E}}(\mathcal{D})$.

Abstract(Pro C^*-algebras)

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An abstract Pro C^*-algebra is a $*$-algebra with family of separating C^*-seminorms.

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Remark

The $*$-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ equipped with the topology defined by the family of C^*-seminorms $p_\alpha(T) = \|T|_{H_\alpha}\|$ is a Pro C^*-algebra.

Abstract (Local operator spaces)

Definition

Let V be a vector space and $\{p_\alpha^n : \alpha \in \Lambda\}$ be a family of separating seminorms for each $n \in \mathbb{N}$ satisfying following properties:

1. $p_\alpha^{n+m}(v \oplus w) \leq \max\{p_\alpha^n(v), p_\alpha^m(w)\}$ for each α
2. $p_\alpha^m(PvQ) \leq \|P\| p_\alpha^n(v) \|Q\|$ for each α

where $v \in M_n(V)$, $w \in M_m(V)$, $P \in M_{m,n}, Q \in M_{n,m}$.

Local operator systems

Definition

Let V be a $*$-vector space consisting of downward filtered family of cones $\{\mathcal{C}_\alpha : \alpha \in \Gamma\}$ satisfying two properties $\mathcal{C}_\alpha$ is a cone (need not be proper) in V where $\mathcal{C}_\alpha \subseteq V_h = \{v \in V : v^* = v\}$ and $\bigcap_\alpha (\mathcal{C}_\alpha \cap -\mathcal{C}_\alpha) = \{0\}$.

Then V is called *local $*$-ordered vector space* and the elements of $\mathcal{C}_\alpha$ are called local positive elements, denoted by $v \geq_\alpha 0$. Also, we write $v_1 \geq_\alpha v_2$ if $v_1 - v_2 \geq_\alpha 0$ in V.

Definition

For a local $*$-ordered vector space $(V, \{\mathcal{C}_\alpha : \alpha \in \Gamma\})$, an element $e \in V_h$ is called *an ordered unit for V* if for all $v \in V_h$ and for every $\alpha \in \Gamma$ there exists $r_\alpha > 0$ such that $r_\alpha e \geq_\alpha v$. If, in addition whenever $re + v \geq_\alpha 0$ for all $r > 0$ and $\alpha \in \Gamma$ and $v \in V_h$ implies $v \in C_\alpha$ we call e is an Archimedean order unit and the triple $(V, \{\mathcal{C}_\alpha : \alpha \in \Gamma\}, e)$ *an Archimedean local $*$-ordered vector space* or in short *A.L.O.U space*.

Definition

Let V be a $*$-vector space. We say that the family $\{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty \ : \ \alpha \in \Gamma\}$ is a *local matrix ordering on V* if

1. $(M_n(V), \{\mathcal{C}_\alpha^n : \alpha \in \Gamma\})$ is a local $*$-ordered vector space for each $n \in \mathbb{N}$,
2. for each $n, m \in \mathbb{N}$ and $X \in M_{n,m}$ and all α, we have that $X^* \mathcal{C}_\alpha^n X \subseteq \mathcal{C}_\alpha^m$.

In this case, we call $(V, \{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty \ : \ \alpha \in \Gamma\})$ a local matrix $*$-ordered vector space.

For $e \in V_h$, let $e_n = \mathrm{diag}(e, e, \ldots, e)$ be the corresponding diagonal matrix in $M_n(V)$. We say that e is *a matrix order unit for V* if e_n is an order unit for $(M_n(V), \{\{\mathcal{C}_\alpha^n \ : \ \alpha \in \Gamma\}\})$ for each n. We say that e is an *Archimedean matrix order unit* if e_n is an Archimedean order unit for $(M_n(V), \{\{\mathcal{C}_\alpha^n \ : \ \alpha \in \Gamma\}\})$ for each n.

Definition

An *abstract local operator system* is a triple $(V, \{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty; \alpha \in \Gamma\}, e)$, where V is a local $*$-ordered vector space, $\{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty : \alpha \in \Gamma\}\}$ is a local matrix ordering on V and e is an Archimedean matrix order unit.

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Remark

Every concrete local operator system is abstract local operator system.
Proof: Let V be concrete local operator system, so there exists H a Hilbert space and a quantized domain in H of upward filtered family $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Gamma}$ of closed subspaces in H whose union $\mathcal{D}=\cup H_\alpha$ is dense in H. Here we have family of cones as $\mathcal{C}_\alpha=\{T \in C_\mathcal{E}^*(\mathcal{D}) \cap V : T|_{H_\alpha} \geq 0\}$.

Examples

1. Every operator system is a local operator system.
2. Consider the set $C(\mathbb{R})$, the space of all complex valued continuous functions defined on $\mathbb{R}$.For every compact subset K of $\mathbb{R}$, we define a family of cones as $C_K = \{f \in C(\mathbb{R})_h : f(x) \geq 0 \forall x \in K\}$ and Archimedean matrix order unit is I(x)=1 $\forall x \in \mathbb{R}$, then $(C(\mathbb{R}), \{\{C_K^n\} : K \subseteq \mathbb{R}, K \text{is compact}\}, I)$ is a local operator system.
3. Let H be Hilbert space, $\mathcal{D}$ is dense subspace in H. Take $T \in C^*_{\mathcal{E}}(\mathcal{D})$ we have $\mathcal{LOS}(T) = \mathrm{span}\{I, T, T^*\}$ is a local operator subsystem of $C^*_{\mathcal{E}}(\mathcal{D})$.

Remark

- *Every local operator system is projective limit of operator systems.*

Proposition

Let V be a local operator system with family of cones $\{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty : \alpha \in \Gamma\}\}$ and e is Archimedean matrix order unit and for each $X \in M_n(V)$ set

$$\|X\|_\alpha^n = \inf \left\{ r \geq 0 : \begin{pmatrix} re_n & X \\ X^* & re_n \end{pmatrix} \in \mathcal{C}_\alpha^{2n} \right\},$$

then $\|\cdot\|_\alpha^n$ is a separating family of $*$-seminorms on $M_n(V)$ and $\mathcal{C}_\alpha^n$ is a closed subset of $M_n(V)$ in the topology induced by this separating family of $*$-seminorms. Hence, $\{V, \{\|\cdot\|_\alpha^n\}_{n=1}^\infty\}$ is a local operator space.

Definition

Let V and W be two abstract local operator systems with $\{C_\alpha : \alpha \in \Gamma\}$ and $\{D_\beta : \beta \in \Omega\}$ family of cones, respectively. A linear map Φ: $V \longrightarrow W$ is called *local positive* if for each $\beta \in \Omega$ there corresponds $\alpha \in \Gamma$ such that $\Phi(C_\alpha) \subseteq D_\beta$. If in addition, $\Phi(e) = f$ where e and f are the Archimedean local matrix units of V and W, respectively then Φ will be called *unital local positive*. Moreover, if Φ is *unital local positive* at each matrix level, we call it *unital local completely positive map*, in short ULCP.

Definition

A linear map Φ: $V \longrightarrow W$ is called local order isomorphism if Φ is bijective, $\Gamma = \Omega$ and $\Phi(C_\alpha) = D_\alpha$ for all $\alpha \in \Gamma$.

Theorem (Representation theorem)

Let V be an abstract local operator system, then there exists a unital complete local order embedding Φ from V into $C^*_{\mathcal{E}}(D)$. Hence abstract local operator systems are equivalent to concrete local operator systems.

LOMIN and LOMAX structures

- Let $(V, \{V_\alpha^+ : \alpha \in \Gamma\}, e)$ be an Archimedean local order unit space. For each $n \in \mathbb{N}$ and each $\alpha \in \Gamma$, define

$$(\mathcal{C}_\alpha^n)^{\min}(V) := \{(v_{ij}) \in M_n(V) \ : \ \sum_{i,j=1}^{n} \bar{\lambda}_i \lambda_j v_{ij} \in \mathcal{C}_\alpha \forall \lambda_1, ...\lambda_n \in \mathbb{C}.\}$$

Definition

Let $(V, \{\mathcal{C}_\alpha : \alpha \in \Gamma\}, e)$ be an Archimedean local ordered unit space. We define $LOMIN(V)$ to be the local operator system $\left(V, \left\{\{(\mathcal{C}_\alpha^n)^{\min}(V)\}_{n=1}^{\infty} \ : \ \alpha \in \Gamma\right\}, e\right)$.

Remark

Let $(V, \{V_\alpha^+\}{:}\alpha \in \Gamma, e)$ be an A.L.O.U space. If $(V, \left\{\{\mathcal{C}_\alpha^n\}_{n=1}^{\infty} \ : \ \alpha \in \Gamma\right\}, e)$ is any local operator system on V with $\mathcal{C}_\alpha^1 = V_\alpha^+$, for each α, then $\mathcal{C}_\alpha^n \subseteq (\mathcal{C}_\alpha^n)^{\min}(V)$ for all n and all α. Thus $LOMIN(V)$ is the weakest local operator system.

Cont.

- Let $(V, \{V_\alpha^+ : \alpha \in \Gamma\}, e)$ be a local order $*$-vector space. Define $(\mathcal{D}_\alpha^n)^{\max}(V)=\{\sum_{i=1}^{k} a_i \otimes v_i;\ v_i \in V_\alpha^+, a_i \in M_n^+, i = 1, 2, ..., k; k \in \mathbb{N}\}$ and $D_\alpha^{\max}(V)=\{(\mathcal{D}_\alpha^n)^{\max}(V)\}_{n=1}^{\infty}$ for each α.

Proposition

Let $(V, \{V_\alpha^+ : \alpha \in \Gamma\}, e)$ be an Archimedean local order unit space, then the cones $(\mathcal{D}_\alpha^n)^{\max}(V)$ are given by

$$(\mathcal{D}_\alpha^n)^{\max}(V) = \{\gamma \mathrm{diag}(v_1, v_2, \ldots, v_m)\gamma^* \ :\ \gamma \in M_{n,m}, v_i \in V_\alpha^+, i = 1, 2, \ldots, m; m \in \mathbb{N}$$

Remark

With the above cones; we get the strongest local operator system $LOMAX(V)$.

Tensor products

Definition

Given local operator systems $(V, \{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty : \alpha \in \Gamma\}, e_V)$ and $\left(W, \left\{\{\mathcal{D}_\beta^n\}_{n=1}^\infty : \beta \in \Lambda\right\}, e_W\right)$, a local operator system structure $l\mathcal{T}$ on $V \otimes W$ is a matricial cone structure given by $\left\{\{\mathcal{T}_\gamma^n\}_{n=1}^\infty : \gamma \in \Omega\right\}$ where $\Omega \cong \Gamma \times \Lambda$ such that:

- $\left(V \otimes W, \left\{\{\mathcal{T}_\gamma^n\}_{n=1}^\infty : \gamma \in \Omega\}\right\}, e_V \otimes e_W\right)$ is a local operator system.
- For every $\alpha \in \Gamma$ and $\beta \in \Lambda$, there exists a $\gamma \in \Omega$ such that $\mathcal{C}_\alpha^n \otimes \mathcal{D}_\beta^m \subseteq \mathcal{T}_\gamma^{nm}$ for all $n, m \in \mathbb{N}$ and for every $\gamma \in \Omega$, there exist $\alpha \in \Gamma$ and $\beta \in \Lambda$ such that $\mathcal{C}_\alpha^n \otimes \mathcal{D}_\beta^m \subseteq \mathcal{T}_\gamma^{nm}$ for all $n, m \in \mathbb{N}$.
- If $\phi \in ULCP(V, M_n)$ and $\psi \in ULCP(W, M_m)$ w.r.t $\mathcal{C}_\alpha$ and $\mathcal{D}_\beta$ respectively, then $\phi \otimes \psi \in ULCP(V \otimes W, M_{nm})$ w.r.t $\mathcal{T}_{(\alpha,\beta)}$ for all $n, m \in \mathbb{N}$.

Cont.

Let τ_1 and τ_2 be two local operator system structure on $S \otimes T$. τ_1 is greater than τ_2 if the identity map from $S \otimes_{\tau_1} T$ to $S \otimes_{\tau_2} T$ is local completely positive map.

- $l\tau$ is *functorial* if for any four local operator systems V_1, V_2, W_1, W_2; $\phi \in ULCP(V_1, V_2)$ and $\psi \in ULCP(W_1, W_2)$ implies the linear map $\phi \otimes \psi : V_1 \otimes W_1 \to V_2 \otimes W_2$ belongs to $ULCP(V_1 \otimes_{l\tau} W_1, V_2 \otimes_{l\tau} W_2)$
- $l\tau$ is *a symmetric* if the map $\theta : v \otimes w \to w \otimes v$ extends to a unital local complete order isomorphism from $V \otimes_{l\tau} W$ onto $W \otimes_{l\tau} V$
- Given local operator systems $V_1 \subseteq V_2$ and $W_1 \subseteq W_2$, if the inclusion map $V_1 \otimes_{l\tau} W_1 \subseteq V_2 \otimes_{l\tau} W_2$ is a local complete order isomorphism onto its range then $l\tau$ is *injective local operator system tensor product.*

Proposition

Let V and W be two local operator systems such that $V = \varprojlim V_\alpha$ and $W = \varprojlim W_\beta$. Then corresponding to any operator system tensor product η, we have a local tensor product η_l such that $V \otimes_{\eta_l} W = \varprojlim V_\alpha \otimes_\eta W_\beta$.

Minimal tensor product

Let $(V, \{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty \ : \ \alpha \in \Gamma\}, e_V)$ and $\left(W, \left\{\{\mathcal{D}_\beta^n\}_{n=1}^\infty \ : \ \beta \in \Lambda\right\}, e_W\right)$ be local operator systems. For each $\alpha \in \Gamma, \beta \in \Lambda, n \in \mathbb{N}$, define

$$\mathcal{T}_{(\alpha,\beta)}^{n(lmin)} := \Big\{(p_{ij}) \in M_n(V \otimes W) : ((\phi \otimes \psi)(p_{ij})) \in M_{nkm}^+, \text{for all } \phi : V \to M_k \text{ and } \psi : W \to M_m, \text{ unital local completely positive maps w.r.t. cones } \mathcal{C}_\alpha \text{ and } \mathcal{D}_\beta \text{ resp. for all } k, m \in \mathbb{N}\Big\}$$

Definition

We call $\left(V \otimes W, \left\{\{\mathcal{T}_{(\alpha,\beta)}^{n(lmin)}\}_{n=1}^\infty : (\alpha, \beta) \in \Gamma \times \Lambda\right\}, e_V \otimes e_W\right)$ *the minimal local tensor product* of V and W and denote it by $V \otimes_{lmin} W$.

Theorem

The mapping lmin: $\mathcal{LO} \times \mathcal{LO} \to \mathcal{LO}$ *sending* (V, W) *to* $V \otimes_{lmin} W$ *is an injective, associative, symmetric, functorial, minimal local operator system tensor product.*

Remark

$V \otimes_{lmin} W = \varprojlim V_\alpha \otimes_{min} W_\beta = V \otimes_{min_l} W$

Maximal tensor product

Let $(V, \{\{\mathcal{C}_\alpha^n\}_{n=1}^\infty : \alpha \in \Gamma\}, e_V)$ and $\left(W, \left\{\{\mathcal{D}_\beta^n\}_{n=1}^\infty : \beta \in \Lambda\right\}, e_W\right)$ be two local operator systems. For each $n \in \mathbb{N}$ and $(\alpha, \beta) \in \Gamma \times \Lambda$, define

$$\mathcal{K}_{(\alpha,\beta)}^{n(lmax)} := \{L(P \otimes Q)L^* : P \in \mathcal{C}_\alpha^k \text{ and } Q \in \mathcal{D}_\beta^m, L \in M_{n,km}, k, m \in \mathbb{N}\}.$$

Definition

We call the local operator system $\left(V \otimes W, \left\{\{\mathcal{T}_{(\alpha,\beta)}^{n(\mathrm{lmax})}\}_{n=1}^\infty : (\alpha, \beta) \in \Gamma \times \Lambda\right\}, e_V \otimes e_W\right)$ the maximal local operator system tensor product of V and W and denote it by $V \otimes_{\mathrm{lmax}} W$.

Theorem

The mapping $\mathrm{lmax} : \mathcal{LO} \times \mathcal{LO} \to \mathcal{LO}$ *sending* (V, W) *to* $V \otimes_{\mathrm{lmax}} W$ *is a symmetric, associative, functorial, maximal local operator system tensor product.*

Remark

$V \otimes_{lmax} W = \underleftarrow{lim} V_\alpha \otimes_{max} W_\beta = V \otimes_{max_l} W$

- For local operator system tensor products $\mathrm{l}\eta$ and $\mathrm{l}\gamma$, V is $(\mathrm{l}\eta, \mathrm{l}\gamma)$-nuclear if the identity map between $V \otimes_{\mathrm{l}\eta} W$ and $V \otimes_{\mathrm{l}\gamma} W$ is a local complete order isomorphism for every local operator system W.

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Thankyou for your kind attention!

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