# An improved calculation of the mass for the resonant spring pendulum

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When the appropriate mass is used to oscillate a spring, the vertical oscillations couple to a pendular swing. Previous calculations of various aspects of this resonance assumed a massless spring as a simple pendulum. This paper improves the estimate of the mass necessary to induce this resonance by describing a massive spring as a physical pendulum and obtains an expression for the mass in terms of the spring constant and various lengths associated with the spring. Several approximations will be considered to simplify the complete expression. Comparisons of the predictive power of these expressions are made for various values of the spring constants. An Appendix discusses the assumption of uniform coil density of a hanging massive spring. © *2004 American Association of Physics Teachers.*
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## I. RESONANT SPRING PENDULUM

Several authors[1–4] have discussed the resonance in which a vertically oscillating spring spontaneously oscillates between spring-bouncing and pendular-swinging. These papers assume massless springs as simple pendula to solve the equations of motion. The goal of this paper is to introduce massive springs as physical pendula in order to more accurately predict the mass that leads to this resonance. For details about the coupling of the modes, the resonance, the parametric instability, and the period of oscillation between these modes, the reader should refer to Refs. 1–4 and references therein. In brief, Olsson[1] cautions that because this system is not linear, superposition is not applicable and, therefore, "the general motion cannot be expressed as a combination of normal modes." He also notes that the "... resonance effect is more correctly known as an autoparametric resonance... " due to the lack of an explicit time dependence in the differential equation. More importantly, he gives a more detailed description of the equations of motion that describe this resonance. Lai[2] solves Olsson's equations and discusses why the conversion between the oscillation modes does not merely occur, but recurs.[5]

A spring with length $\ell$ will oscillate vertically according to the equation of motion:

$$m\ddot{z} + k(z - \ell) = mg, \tag{1}$$

where $z$ is the vertical position with $z = 0$ at the top of the unstretched spring and positive downward, $\ddot{z}$ is the second derivative in time, $m$ is the mass attached to the spring, $k$ is the spring constant, and $g$ is the magnitude of the local gravitational field. Equation (1) is solved by a trigonometric function plus a constant stretch $z(t) = A\cos(\omega_s t) + (\ell + mg/k)$. For the initial position $z_0$, the spring oscillates with amplitude $A = z_0 - \ell - mg/k$ about the equilibrium point $\ell + mg/k$. For a massless spring, the angular frequency is $\omega_s = \sqrt{k/m}$.

Whereas Eq. (1) describes the forces acting on a spring, the pendular motion is due to the torques. A pendulum with length $\ell$ oscillates according to the equation of motion:

$$I\ddot{\theta} + mg\ell \sin\theta = 0. \tag{2}$$

For simple pendula, $I = m\ell^2$, and for small angles, $\sin\theta \approx \theta$. Equation (2) also is solved by a trigonometric function $\theta(t) = \Theta_0 \cos(\omega_p t)$, where $\omega_p = \sqrt{mg\ell/I} = \sqrt{g/\ell}$.

When the autoparametric resonance occurs, it is observed that the period of the pendular motion, $T_p$, is equal to twice the spring-like period, $T_s$:

$$2T_s = T_p, \tag{3}$$

where, for a massless spring and a simple pendulum,

$$T_s = 2\pi\sqrt{\frac{m}{k}}, \quad T_p = 2\pi\sqrt{\frac{\ell}{g}}. \tag{4}$$

As mentioned in Ref. 1, Eq. (3) requires that a spring with the unstretched length $\ell_s$ must be stretched to a length $\ell = 4mg/k$ for the resonance to occur. Because a mass $m$ will stretch a spring to the length $\ell = \ell_s + mg/k$, this resonance occurs when

$$m = \frac{k\ell_s}{3g}. \tag{5}$$

Equation (5) assumes a massless spring and gives only roughly approximate values for the various springs that we will consider. The detailed changes due to considering a massive spring as a physical pendulum should allow us to more closely predict the experimental value of the required mass.

The main goal of this paper is to give a more accurate calculation of the required mass, while minimizing the complexity of the final equation. A secondary goal is to help students learn about approximations as well as provide an opportunity to numerically solve a cubic equation that is applicable to an observable phenomenon. This resonance phenomenon makes a good problem for undergraduates for a variety of reasons. The resonance is not necessarily easy to produce in an unfamiliar spring by trial and error because it is difficult to see how close the system is to resonance. Predicting the resonance mass forces the students to reconsider the assumptions (massless springs and simple pendula) in the equations that have been derived in class and probably used in the laboratory.

In Sec. II, I will review how previous authors have introduced the spring mass into the vertical oscillations and com-

bine that with the physical pendulum to derive a cubic equation for the mass that induces the resonance. In Sec. III, I consider three approximations to this cubic equation in order to find an expression that is more convenient. In Sec. IV, I will discuss the experimental apparatus and the theoretical predictions, and compare the predictions to the masses that experimentally induce a resonance in the springs. The conclusion is in Sec. V. An appendix gives more details about the major underlying assumption.

## II. INTRODUCING THE SPRING MASS

In this section, I will introduce the mass of the spring into the vertical spring oscillations and then into the pendulum swing. When we combine these expressions to predict the resonance, we will find that the distribution of mass is relatively important.

### A. Massive springs

The consideration of the correction to the spring oscillations due to including the mass of a spring has led to many papers.[6–14] There is a commonly used correction that employs an approximation that should be reasonably valid for all but the softest springs. For the convenience of the reader, a brief summary will be repeated here with references to the literature for details.

There are two situations in which the mass of the spring should be included. First, the spring mass shifts the equilibrium position of a vertically hanging spring because more mass must be supported by the upper portions of the spring. This static correction will modify $T_p$, the pendulum period, but will not affect $T_s$, the period of the spring-like bouncing motion. Second, the inertia of a massive spring produces a dynamic correction to $T_s$. The corrections due to these effects are not the same and are more prominent in a softer spring.

To see the static effect, consider a spring with mass $m_s$, spring constant $k$, and unstretched length $\ell_0$. Imagine it as a series of $N$ springs, labeled from top to bottom as 1 through $N$, each with mass $m_i = m_s/N$, spring constant $k_i = Nk$, and unstretched length $\ell_{i0} = \ell_0/N$. The $i$th spring supports the $(N-i)$ springs below it as well as the mass at the end and therefore each is stretched to length $\ell_i = \ell_{i0} + mg/k_i + (N-i)m_i g/k_i$. The total length of the spring series is

$$\ell = \sum_{i=0}^{N-1} \ell_i = \sum_{i=0}^{N-1} \left[ \ell_{i0} + \frac{mg}{k_i} + (N-i)\frac{m_i g}{k_i} \right] \quad (6a)$$

$$= \sum_{i=0}^{N-1} \left(\frac{1}{N}\right) \left[ \ell_0 + \frac{mg}{k} + (N-i)\frac{m_s g}{Nk} \right]$$

$$= \ell_0 + \frac{mg}{k} + \left[ N^2 - \frac{N(N+1)}{2} \right] \frac{m_s g}{N^2 k}$$

$$= \ell_0 + \frac{mg}{k} + \frac{m_s g}{2k}. \quad (6b)$$

I have used the fact that $\Sigma_1^N i = N(N+1)/2$ and taken the large $N$ limit so that $1/N \to 0$. In the continuum limit, Eq. (6) can be written as[6]

$$\ell = \int_0^\ell d\ell = \int_0^\ell \frac{1}{\ell} \left( \ell_0 + \frac{mg}{k} + \frac{(\ell - z)m_s}{\ell} \frac{g}{k} \right) dz \quad (7a)$$

$$= \ell_0 + \frac{mg}{k} + \frac{m_s g}{2k}, \quad (7b)$$

where $z$ is measured from the top of the spring and $(\ell - z)m_s/\ell$ is the fraction of the spring below the point $z$. Equation (6) or (7) shows a correction to the static length in which $m \to m + m_s/2$.

Note first that the term $(N-i)m_i g/k_i$ in Eq. (6a), or equivalently $(\ell - z)m_s g/\ell k$ in Eq. (7a), implies that the coils stretch more at the top, where the spring supports more weight. This difference means that the center of mass of the spring will be lower than half-way down the spring, which will be of interest when we discuss the pendular motion. Second, Mak[7] started from Ref. 6 and derived a piecewise function that generalizes the static correction to include the effect of a constant force necessary to initially extend a spring, thereby providing a discrete version of Eq. (20) in Ref. 8 and footnote 16 in Ref. 9. Finally, for convenience, I will consider the freely hanging length,

$$\ell_s = \ell_0 + \frac{m_s g}{2k}, \quad (8)$$

to be a measured quantity rather than measuring $\ell_0$ and adding $m_s g/2k$.

The dynamic correction due to a bouncing, finite-mass spring affects the period of oscillation. This correction can be derived in a deceptively easy manner by using the kinetic[11] and potential energy in Lagrangian mechanics if one assumes uniform stretch, or equivalently, an uniform mass density or uniform coil density. This derivation is deceptively easy because a more careful and complicated treatment gives a transcendental equation (discussed in the following) that reduces to the Lagrangian result in the appropriate limit. Given a spring of length $\ell_0$ and mass $m_s$ and an uniform linear mass density $\lambda = m_s/\ell$, we can find the velocity $v(z)$ of a differential portion of the spring at a distance $z$ measured from the top of the coils: $v(z) = v_m z/\ell$, where $v_m$ is the velocity of the added mass and $\ell$ is given by Eq. (6). The kinetic energy of the spring between $z$ and $z + dz$ is

$$dK = \frac{1}{2}\lambda [v(z)]^2 dz = \frac{1}{2} \frac{\lambda v_m^2 z^2}{\ell^2} dz. \quad (9)$$

The total kinetic energy of the spring is the integral of Eq. (9) plus that of the mass hanging from the spring:

$$K = \frac{1}{2}mv_m^2 + \int_0^\ell \frac{1}{2} \frac{\lambda v_m^2 z^2}{\ell^2} dz = \frac{1}{2}\left(m + \frac{m_s}{3}\right)v_m^2. \quad (10)$$

Equation (10) shows a correction to the dynamic length in which $m \to m + m_s/3$. When hung, the bottom of the unloaded spring sits at $z = \ell_0 + m_s g/2k$. The spring potential energy plus the gravitational potential energy is

$$U(z) = \frac{1}{2}k(\ell_0 - z)^2 - mgz - m_s g \frac{z}{2}. \quad (11)$$

We have again assumed uniform mass density to write the $m_s$ term.

The kinetic and potential energies, Eqs. (10) and (11), can be used in the Lagrangian formulation to derive the new equation of motion. The generalization of Eq. (1) is

$$\left(m+\frac{m_s}{3}\right)\ddot{z}+kz=k\left[\ell_0+\left(m+\frac{m_s}{2}\right)\frac{g}{k}\right],\quad(12)$$

with the solution

$$z(t)=(z_0-\ell)\cos(\omega_s t)+\ell,\quad(13)$$

where $\omega_s=\sqrt{(m+m_s/3)/k}$ and $\ell=\ell_0+(m+m_s/2)g/k$ as in Eqs. (6) and (7). [A shift of the coordinate system by $\ell=\ell_0+(m+m_s/2)g/k$ makes Eq. (13) look simpler, but hides the shift in the equilibrium position.] It follows that the period of oscillation becomes

$$T_s=2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}.\quad(14)$$

The $m_s/3$ in Eq. (14) is the dynamic correction and may be contrasted with the $m_s/2$ equilibrium shift of Eq. (7), the static correction. Students who have not seen Lagrangians can infer the dynamic correction to the period from Eq. (10), because it is reasonable to expect the period to see the same mass as the kinetic energy, $(m+m_s/3)$, rather than as the equilibrium shift, $(m+m_s/2)$.

Several papers[6–10,12–14] have been published on these corrections due to the effect of the spring mass. Those authors[6–10] that consider both corrections make a point of distinguishing the static correction from the dynamic correction. As alluded to earlier, a careful treatment of the dynamic spring in terms of the position of the spring shows that including the mass of the spring leads to a transcendental equation for the period.

In an early paper, Weinstock[10] considered a mass in uniform circular motion which stretches the spring the same amount as a hanging mass and introduced an $m_s/3$ correction to the mass dependence of the frequency. He also considered the mass oscillating about its equilibrium radius and found that the angular frequency satisfies the transcendental equation, which in my notation reads

$$\frac{2\pi\sqrt{m_s/k}}{T_s}\tan\left(\frac{2\pi\sqrt{m_s/k}}{T_s}\right)=\frac{m_s}{m}.\quad(15)$$

Equation (15) reduces to Eq. (14), with dynamic correction $m+m_s/3$, in the small $m_s/m$ limit.

Heard and Newby[8] and Cushing[9] also consider both the static and dynamic corrections. For the dynamic spring, they solved the appropriate differential equations. Reference 8 went a step further by considering a vertical soft spring to find that in the $m_s/m\rightarrow\infty$ limit, $T_s\rightarrow[4/(2n+1)]\sqrt{m_s/k}$, where $n$ is any positive integer (corresponding to the periodic nature of trigonometric functions). These periods correspond to the eigenvalues that solve the appropriate differential equation given in their paper. These periods also solve Eq. (15) in the large $m_s/m$ limit. Consequently, as discussed by Galloni and Kohen,[6] the dynamical correction in Eq. (14) is different in the large $m_s/m$ limit than in the small $m_s/m$ limit. Equation (14) is given in Ref. 6 as

$$T=2\pi\sqrt{\frac{1}{k}\left(m+\frac{m_s}{D}\right)},\quad(16)$$

where $D$ is given by the solution to their version of Eq. (15) with the limiting behavior

$$\lim_{m_s/m\rightarrow0}D=3,\qquad\lim_{m_s/m\rightarrow\infty}D=\frac{\pi^2}{4}.\quad(17)$$

The dynamical correction is discussed further in Refs. 12, 13, 14, and 9. Three methods for solving Eq. (15) are given in Weinstock's later paper,[12] in which it was shown that the lowest frequency mode is sufficient to describe the small $m_s/m$ limit. The uniform density approximation was used to show that the $m_s/3$ dynamic correction is reasonable up to $m_s=m$. McDonald[13] showed that a closed form for $D$ exists for the conical spring, reproduced the result for $D$ in Ref. 6, and compared $m_s/D$ to the equivalent correction term for conical springs. Bowen[14] considered an unloaded slinky with general initial conditions for the nonfixed end, emphasizing the $\pi^2/4$ result. Cushing[9] showed explicitly that the small $m_s/m$ case necessarily transitions to the large $m_s/m$ case so that the lowest normal mode is always the dominant one.

It should be emphasized that adding a term $m_s/3$ as in Eqs. (10) and (14) is based on the assumption of uniform stretch ($\lambda=m_s/l$ is a uniform coil density). This assumption becomes less applicable for softer springs and depends on the ratio $m_s/m$, as seen in Eq. (17). In most cases (all but the smaller $k$ springs cited in Refs. 7 and 9), it seems sufficient to use $m_s/3$ as was done here. The relevant results are discussed in Appendix B.

## B. Summary of the physical pendulum

As stated following Eq. (2), $\omega_p=\sqrt{mg\ell/I}$. For a physical pendulum, we cannot use the simplification $I=m\ell^2$, which gives $T_p$ in Eq. (4). Rather, we must express the period more generally as

$$T_p=2\pi\sqrt{\frac{I}{mg\ell}},\quad(18)$$

where the denominator is due to the torques that drive the oscillation. To see how this general expression changes with different assumptions, consider three cases. Case 1: A simple pendulum with a massless string has $I=m_{\text{bob}}\ell^2$. If we use this value of $I$ and that the torque on the bob is $m_{\text{bob}}g\ell$, we have $T=2\pi\sqrt{\ell/g}$. Case 2: A solid rod pendulum has $I=\frac{1}{3}m_{\text{rod}}\ell^2$ and, because the torque acts at the center of mass of the rod, the denominator of Eq. (18) becomes $m_{\text{rod}}g(\ell/2)$. The period of the rod is then

$$T=2\pi\sqrt{\frac{\frac{1}{3}m\ell^2}{mg(\ell/2)}}=2\pi\sqrt{\frac{2\ell}{3g}}.\quad(19)$$

Case 3: A massive support with a massive bob, like the case of interest, combines these two cases by separately adding the moments of inertia in the numerator, $I=m_{\text{bob}}\ell^2+\frac{1}{3}m_{\text{rod}}\ell^2$, and the torques in the denominator, $\tau=m_{\text{bob}}g\ell+m_{\text{rod}}g(\ell/2)$, giving a period of

$$T=2\pi\sqrt{\frac{(m_{\text{bob}}+\frac{1}{3}m_{\text{rod}})\ell}{(m_{\text{bob}}+\frac{1}{2}m_{\text{rod}})g}}.\quad(20)$$

Note that Eq. (20) reduces to either the massless string or the solid rod pendulum when the appropriate mass is set to zero.

It is an interesting coincidence that the moment of inertia sees the same fraction of $m_s$ as the kinetic energy discussed in Sec. II A and that the torque sees the same fraction of $m_s$ as the stretching of the spring. However, the mathematical expression for the moment of inertia of the spring pendulum will be further complicated by the length dependence as discussed in Sec. II C. A careful treatment of the length dependence in Eq. (18), specifically in terms of the mass distribution, shows that the length does not cancel as simply as it did in Eq. (20).

## C. Massive spring as a physical pendulum

Now that we have introduced the mass into the bouncing spring and swinging pendulum, we can use Eqs. (14) and (18) to derive an expression for the mass that produces the desired resonance between bouncing and swinging. As discussed in Sec. II B, the distribution of mass in the system must be understood in order to express the period of the pendular motion. To account for the geometry of the swinging spring system, Fig. 1(a) shows the various relevant lengths: the length $\ell_s$ of the spring with no added mass, the amount of stretch $mg/k$ due to the added mass, the distance $\ell_1$ from the top of the coils of the spring to the pivot point, the distance $\ell_2$ from the bottom of the coils of the spring to the top of the added mass, and the distance $\ell_m$ from the top of the added mass to the center of mass. The initial length of the hanging spring, $\ell_s$, is stretched only due to its own weight as given by Eq. (8), and $\ell_2$ and $\ell_m$ depend on the amount of mass added. Figure 1(b) distinguishes the theoretically convenient lengths, $\ell_2$ and $\ell_m$, from the experimentally convenient heights: the distance $h_s$ from the coils to the hook, the height $h_h$ of the hanger, and the height $h_m$ of the masses. These lengths are related according to $\ell_2 = h_s + h_h - h_m$ and $\ell_m = h_m/2$. For simplicity, I will assume that $\ell_2$ and $\ell_m$ are independent of the amount of added mass. Indeed, one might expect that $\ell_m$ is significantly less than $\ell_s$ so that the estimated value of $\ell_m$ is irrelevant as long as it is close enough to its true value. As a rough approximation, the appropriate $\ell_2$ and $\ell_m$ can be estimated by measuring $h_m$ for the mass given by Eq. (5). I will discuss this issue in Sec. IV.

The expressions for the period of a swinging coil can be simplified by assuming that, when swinging, the spring is a rigid rod with mass $m_s$, length $(\ell_s + mg/k)$, and center of mass $[\ell_1 + (\ell_s + mg/k)/2]$. A more accurate, but much less convenient expression for the center of mass is given in Appendix B. The moment of inertia in the numerator of Eq. (18)

can then be found and is due to both the spring and the added mass. It also is a combination of rotations about the center of mass, $I_{c.m.}$, and the effect of the parallel axis theorem $I_{pat}$:

$$I = I_{s,c.m.} + I_{s,pat} + I_{m,c.m.} + I_{m,pat}. \tag{21}$$

The first two terms can be written as

$$I_{s,c.m.} + I_{s,pat} = \frac{1}{12} m_s \left( \ell_s + \frac{mg}{k} \right)^2 + m_s \left( \ell_1 + \frac{\ell_s + \frac{mg}{k}}{2} \right)^2. \tag{22}$$

The first term is the moment of inertia of a slender rod (the spring) rotating about its center of mass. If $\ell_1 = 0$, the two terms in Eq. (22) would combine to give $\frac{1}{3} m_s (\ell_s + mg/k)^2$, the moment of inertia of a slender rod about one end. The definition,

$$x_1 \equiv \frac{\ell_1}{\ell_s + mg/k}, \tag{23}$$

leads to the expression

$$I_{s,c.m.} + I_{s,pat} = m_s \left( \ell_s + \frac{mg}{k} \right)^2 \left( \frac{1}{3} + x_1 + x_1^2 \right). \tag{24}$$

The third term in Eq. (21) is

$$I_{m,c.m.} = \frac{1}{12} m (h_m)^2 = \frac{1}{12} m (2\ell_m)^2 = \frac{1}{3} m \ell_m^2. \tag{25}$$

The length $2\ell_m$ is used because $\ell_m$ is half the height of the added mass, $h_m$. The fourth term in Eq. (21) describes the mass at the end of all of the lengths:

$$I_{m,pat} = m \left( \ell_1 + \ell_s + \frac{mg}{k} + \ell_2 + \ell_m \right)^2. \tag{26}$$

Finally, the denominator of Eq. (18) can be expressed as

$$Mg\ell \rightarrow mg \left[ \ell_1 + \ell_s + \frac{mg}{k} + \ell_2 + \ell_m \right] + m_s g \left[ \ell_1 + \frac{\left( \ell_s + \frac{mg}{k} \right)}{2} \right]. \tag{27}$$

If we substitute Eqs. (24)–(27) into Eq. (18) and combine this result with $T_s$ from Eq. (14) as expressed by the square of Eq. (3), we find

$$4 \frac{m + \frac{m_s}{3}}{k} = \frac{m_s \left( \ell_s + \frac{mg}{k} \right)^2 \left[ \frac{1}{3} + x_1 + x_1^2 \right] + \frac{1}{3} m \ell_m^2 + m \left[ \left( \ell_s + \frac{mg}{k} \right) + \ell_1 + \ell_2 + \ell_m \right]^2}{mg \left[ \left( \ell_s + \frac{mg}{k} \right) + \ell_1 + \ell_2 + \ell_m \right] + \frac{1}{2} m_s g \left( \ell_s + \frac{mg}{k} + 2\ell_1 \right)}. \tag{28}$$

The complicated expression for $m$ in Eq. (28) explains why it is not usually considered in this form. Note that if $\ell_1 = \ell_2 = \ell_m = 0$, then the substitution of Eqs. (24)–(27) into Eq. (18) for $T_p$ reproduces Eq. (20). This case will be useful

in Sec. III D. For compactness, we have written only one of the contributions in Eq. (28) in terms of $x_1$. If we gather terms in powers of $m$ without using the $x_1$ notation, we obtain

(a)



Bottom of Spring
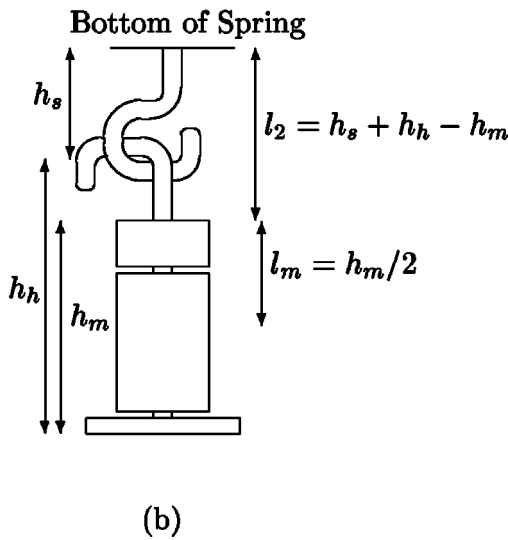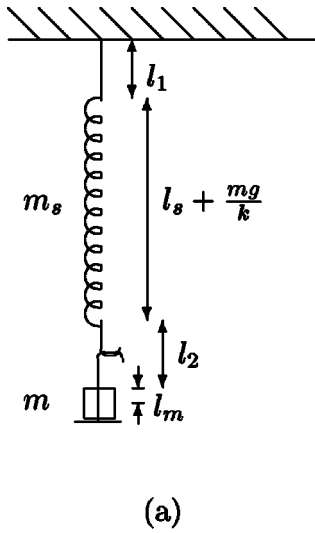
$$l_2 = h_s + h_h - h_m$$

$$l_m = h_m/2$$

(b)

Fig. 1. The hanging spring can be separated into a variety of distinct measurements. (a) The theoretically useful quantities; $\ell_s$ is the length of the hanging spring with no mass attached but stretched due to its own mass; $\ell_m$ is the distance to the center of mass for the mass added. The total length is denoted as $\ell = \ell_1 + \ell_s + mg/k + \ell_2 + \ell_m$. (b) A close up of the added mass to relate the theoretically useful quantities ($\ell$) to the experimentally easy-to-measure quantities ($h_h$).

$$0 = 3m^3 + \left[ 3m_s + 2\frac{k}{g}(\ell_s + \ell_1 + \ell_2 + \ell_m) \right] m^2$$
$$+ \left[ \frac{2}{3}m_s^2 + \frac{k}{g}\left( \frac{8}{3}\ell_s + \frac{13}{3}\ell_1 + \ell_2 + \ell_m \right) m_s \right.$$
$$\left. - \frac{1}{3}\left( \frac{k\ell_m}{g} \right)^2 - \left( \frac{k}{g} \right)^2 (\ell_s + \ell_1 + \ell_2 + \ell_m)^2 \right] m$$
$$+ \left[ \frac{2}{3}m_s^2 \frac{k}{g}(\ell_s + 2\ell_m) \right.$$
$$\left. - m_s \left( \frac{k}{g} \right)^2 \left( \frac{1}{3}\ell_s^2 + \ell_s\ell_1 + \ell_1^2 \right) \right]. \quad (29)$$

Given the mass $m_s$ of the spring, the spring constant $k$, and

the various lengths appropriate to the spring, the coefficients can be included in a root-finding program to solve for what should be the best prediction of the mass $m$ that will cause the spring to excite the autoparametric resonance.

We have made three assumptions to obtain Eq. (29). We assumed that the spring is a rigid rod for calculating the moment of inertia $I_s$, that $\ell_2$ and $\ell_m$ are independent of the mass, and that the spring has uniform density giving both $(m + m_s/3)$ and the expression for the center of mass in Eqs. (22) and (27). The first assumption is reasonable because when the appropriate mass produces the autoparametric resonance between swinging and bouncing, the few oscillations in the purely swing-mode do not have any visible bounce, that is, the swinging spring behaves as a rigid rod. The second assumption will be reasonable if and only if $h_m$ can be estimated to sufficient precision. The details of generalizing the third assumption are relegated to Appendix B.

## III. APPROXIMATIONS TO THE CUBIC

Solving the cubic polynomial in Eq. (29) for the added mass $m$ is straightforward using standard numerical root-finding techniques. However, reasonable approximations should simplify Eq. (28) considerably. In this section, I will introduce three seemingly reasonable approximations to find a more useful approximation to Eq. (28).

We will consider the case $\ell_1 = \ell_2 = \ell_m = 0$ in Sec. III A and then $\ell_m \neq 0$ in Sec. III B. We build on these approximations in Sec. III C where we keep all three lengths and Sec. III D where we build a mathematically convenient approximation.

### A. The long-spring approximation

When $x_1$ is set to zero, Eq. (28) remains cubic in $m$ due to its dependence on $\ell_2$ and $\ell_m$. In the long-spring approximation, we set $\ell_1$ (and $x_1$) $= \ell_2 = \ell_m = 0$, and Eq. (28) becomes

$$4\frac{m + \frac{m_s}{3}}{k}\left[ mg\left( \ell_s + \frac{mg}{k} \right) + \frac{1}{2}m_s g\left( \ell_s + \frac{mg}{k} \right) \right]$$
$$= \frac{1}{3}m_s\left( \ell_s + \frac{mg}{k} \right)^2 + m\left( \ell_s + \frac{mg}{k} \right)^2. \quad (30)$$

We can cancel $(\ell_s + mg/k)$ and $(m + m_s/3)$, giving an expression linear in $m$,

$$m = \frac{k\,\ell_s}{3g} - \frac{2m_s}{3}, \quad (31)$$

which is a minor change from the massless spring simple pendulum of Eq. (5).

### B. Weakening the long-spring approximation

Let us take $\ell_m \neq 0$ and $\ell_1 = \ell_2 = 0$, because $\ell_1$ and $\ell_2$ are always added to $\ell_s$, whereas $\ell_m$ is not added to a larger quantity in the $\frac{1}{3}m\ell_m^2$ term. With this choice, Eq. (28) becomes

$$4\,\frac{m+\dfrac{m_s}{3}}{k}$$

$$=\frac{\dfrac{1}{3}m_s\left(\ell_s+\dfrac{mg}{k}\right)^2+\dfrac{1}{3}m\ell_m^2+m\left[\left(\ell_s+\dfrac{mg}{k}\right)+\ell_m\right]^2}{mg\left[\left(\ell_s+\dfrac{mg}{k}\right)+\ell_m\right]+\dfrac{1}{2}m_sg\left(\ell_s+\dfrac{mg}{k}\right)}. \tag{32}$$

Because Eq. (32) is still cubic in $m$, I will assume that $\ell_m\ll\ell_s$ and define $x_m$ as

$$x_m\equiv\frac{\ell_m}{\ell_s+\dfrac{mg}{k}}\approx\frac{\ell_m}{\ell_s+\dfrac{k\ell_s}{3g}\dfrac{g}{k}}=\frac{3\ell_m}{4\ell_s}, \tag{33}$$

where $m$ has been replaced by the massless-spring, simple-pendulum approximation of Eq. (5). After we divide the numerator and denominator of the right-hand side of Eq. (32) by $(\ell_s+mg/k)$ and collect terms in powers of $m$, we obtain a quadratic equation for $m$ that can be solved given $\ell_m$, $m_s$, $k$, and $\ell_s$,

$$\left[3+2x_m-\frac{4}{3}x_m^2\right]m^2+\left[\left(3+\frac{4}{3}x_m\right)m_s\right.$$
$$\left.-\left(1+2x_m+\frac{4}{3}x_m^2\right)\frac{k\ell_s}{g}\right]m+\left[\frac{2}{3}m_s^2-\frac{k\ell_sm_s}{3g}\right]=0. \tag{34}$$

Note that Eq. (34) reduces to Eq. (31) in the limit $x_m\to0$.

## C. Collecting small terms

To this point, the theoretical results either are poor predictors of the desired mass $m$, such as Eqs. (5) and (31), or are higher-order expressions, such as Eqs. (29) and (34). Equation (31) will give poor predictions for $m$ because the predictions of Eq. (5) are, in most cases, smaller than the necessary physical mass.

The terms in Eq. (28) do not readily cancel because of the different combinations of $\ell_1$, $\ell_2$, and $\ell_m$. Let us first replace each of these by $\ell_{\text{ave}}$, the average of these three quantities. This replacement reduces Eq. (28), without the $x_1$ notation, to

$$4\,\frac{m+\dfrac{m_s}{3}}{k}=\frac{\dfrac{1}{3}m_s\left[\left(\ell_s+\dfrac{mg}{k}\right)^2+3\ell_{\text{ave}}\left(\ell_s+\dfrac{mg}{k}\right)+3\ell_{\text{ave}}^2\right]+\dfrac{1}{3}m\ell_{\text{ave}}^2+m\left[\ell_s+\dfrac{mg}{k}+3\ell_{\text{ave}}\right]^2}{mg\left[\ell_s+\dfrac{mg}{k}+3\ell_{\text{ave}}\right]+m_sg\dfrac{1}{2}\left[\ell_s+\dfrac{mg}{k}+2\ell_{\text{ave}}\right]}. \tag{35}$$

If we rewrite Eq. (35) in terms of $L\equiv(\ell_s+mg/k+3\ell_{\text{ave}})$ and complete the square in the $m_s$ term, we obtain

$$4\,\frac{m+\dfrac{m_s}{3}}{k}=\frac{\left(m+\dfrac{m_s}{3}\right)L^2-\dfrac{m_s}{3}\left[3\ell_{\text{ave}}\left(\ell_s+\dfrac{mg}{k}\right)+6\ell_{\text{ave}}^2\right]+\dfrac{1}{3}m\ell_{\text{ave}}^2}{\left(m+\dfrac{m_s}{2}\right)gL-\dfrac{1}{2}m_sg\ell_{\text{ave}}}. \tag{36}$$

We define the hopefully small quantity

$$y\equiv\frac{\ell_{\text{ave}}}{L}, \tag{37}$$

divide the numerator and denominator by $L$, and find

$$4\,\frac{m+\dfrac{m_s}{3}}{k}$$

$$=\frac{\left(m+\dfrac{m_s}{3}\right)L-m_sy\left[\left(\ell_s+\dfrac{mg}{k}\right)+2\ell_{\text{ave}}\right]+\dfrac{1}{3}my\ell_{\text{ave}}}{\left(m+\dfrac{m_s}{2}\right)g-\dfrac{1}{2}m_sgy}. \tag{38}$$

We then multiply both sides by $(m+m_s/2)g-\frac{1}{2}m_sgy$, collect terms of order $y$ on the right, and obtain

$$4\frac{g}{k}\left(m+\frac{m_s}{3}\right)\left(m+\frac{m_s}{2}\right)$$

$$=\left(m+\frac{m_s}{3}\right)L$$

$$+y\left[2\left(m+\frac{m_s}{3}\right)\frac{m_sg}{k}-m_s(L-\ell_{\text{ave}})+\frac{m\ell_{\text{ave}}}{3}\right]. \tag{39}$$

If we divide through by $(m+m_s/3)$, solve for $m$, and collect the $y$ terms to $\mathcal{O}(y)$, we find

$$m=\frac{k}{3g}(\ell_s+3\ell_{\text{ave}})-\frac{2}{3}m_s+\mathcal{O}(y). \tag{40}$$

If we ignore terms of order $y$, Eq. (40) gives a better estimate of the mass. Alternatively, given the experimental value for the mass, we can estimate the magnitude of the $\mathcal{O}(y)$ terms and evaluate the approximation $l_{\text{ave}}\ll L$.

Table I. The physical dimensions of the springs used. For all but the two stiffest springs, a mass-hanger with height $h_h = 8.000(4)$ cm was used. The second stiffest spring used a hanger with $h_h = 14.806(4)$ cm. The stiffest spring used one with $h_h = 28.2(1)$ cm. The other lengths can be found from $\ell_2 = h_s + h_h - h_m$ and $\ell_m = h_m/2$. The height of the added mass $h_m$ is given for the mass predicted by the equation indicated. All lengths are in centimeters.

| $k$ (N/m) | $m_s$ (kg) | $\ell_s$ | $\ell_1$ | $h_s$ | Eq. (5) $h_m$ | Eq. (42) $h_m$ | Experimental $h_m$ |
|---|---|---|---|---|---|---|---|
| 3.059(1) | 0.6779(2) | 293.3(1) | 2.00(5) | 2.00(5) | 3.419(4) | N/A | N/A |
| 6.696(3) | 0.3390(2) | 121.4(1) | 0.00 | 2.00(5) | 3.272(4) | 1.346(4) | 1.276(4) |
| 25.16(4) | 0.0829(2) | 26.6(1) | 1.23(5) | 1.32(5) | 2.846(4) | 3.316(4) | 3.456(4) |
| 28.63(5) | 0.0824(2) | 27.6(1) | 1.40(5) | 0.95(5) | 3.292(4) | 3.742(4) | 3.809(4) |
| 36.26(8) | 0.0884(2) | 27.1(1) | 1.30(5) | 1.30(5) | 4.109(4) | 4.535(4) | 4.912(4) |
| 40.9(1) | 0.0846(2) | 27.1(1) | 1.20(5) | 1.20(5) | 4.315(4) | 4.992(4) | 5.518(4) |
| 51.4(2) | 0.0786(2) | 27.2(1) | 1.50(5) | 1.30(5) | 5.012(4) | 7.346(4) | 6.849(4) |
| 1002(2) | 0.0629(2) | 6.7(1) | 1.1(1) | 5.00(5) | 6.35(5) | 20.45(5) | 23.84(5) |

## D. A mathematically convenient approximation

We have seen that the spring-like oscillation is affected by the spring mass by a factor of $(m + m_s/3)$. We also saw that a rod-mass pendulum has the moment of inertia $I = (m + m_s/3)\ell^2$ and torque $\tau = (m + m_s/2)g\ell$ as if each saw the additional mass at the bottom of the spring. Although physically inappropriate, if we simply extend Eq. (20) by including the lengths $\ell_1$, $\ell_2$, and $\ell_m$ and substitute this result into Eq. (3) as before, we find

$$4\,\frac{m + \dfrac{m_s}{3}}{k} = \frac{\left(m + \dfrac{m_s}{3}\right)\left(\ell_1 + \ell_s + \dfrac{mg}{k} + \ell_2 + \ell_m\right)^2}{\left(m + \dfrac{m_s}{2}\right)g\left(\ell_1 + \ell_s + \dfrac{mg}{k} + \ell_2 + \ell_m\right)}. \quad (41)$$

Equation (41) should be compared to Eq. (28) to see how it differs from a more physically accurate treatment. Equation (41) is, however, mathematically convenient and easily reduces to

$$m = \frac{k}{3g}(\ell_s + \ell_1 + \ell_2 + \ell_m) - \frac{2}{3}m_s, \quad (42)$$

which is Eq. (40) without the $\mathcal{O}(y)$ term. The physical interpretation is that if $\ell_1$, $\ell_2$, and $\ell_m$ are sufficiently less than $\ell_s$, they can be handled conveniently as in Eq. (40), but even small values are not so small that they can be set to zero as in Eq. (31).

## IV. ANALYSIS OF THE APPROXIMATIONS

After describing the experimental apparatus and the uncertainty, we will compare the predictions of the various approximations to the mass that actually induces the resonance.

### A. The experiment

Our goal is to find a useful expression for the mass that induces the autoparametric resonance between the vertical oscillations and the pendular swinging of a spring. To analyze the accuracy of the predictions, seven springs were used. The spring constants, masses, and various lengths are given in Table I. The five springs with spring constants ranging from 25 to 52 N/m were selected from a standard undergraduate laboratory set. A wave-demonstration spring was used to test the predictions on a low $k$ spring. By first using the full length of the spring and then clamping it half-way

down, I was able to investigate the results for $k = 3.059$ N/m and $k = 6.696$ N/m. Finally, a spring-scale was removed from its casing to test a high $k$ spring.

The values of $k$ in Table I are not needed because $k$ appears in the ratio $k/g$, which can be measured directly. A mass hanging in equilibrium from a spring will satisfy $kz = mg$; therefore, the slope of the displacement versus the added mass (the independent variable) will give $g/k$. To obtain the values of the spring constant, we assumed $g = 9.80(1)$ m/s$^2$. These values are listed as $k$ in Table I. Appendix A discusses some of the relevant details of this measurement.

While addressing the uniform density issues (raised at the end of Sec. II A), Appendix B discusses an alternative, direct measurement of $k$. Because $k$ can be measured directly, it is possible[15] to combine the measurements of $g/k$ and $k$ to calculate the local gravitational field.

Table I also lists the physical lengths of the equipment. The length $\ell_s$ was measured with a meter stick. The lengths $\ell_1$ and $h_s$ were measured with a vernier caliper, which has a precision of 0.002 cm. To account for slight awkwardness in measuring as well as for the possibility that the support ring might stretch when weight was added, this uncertainty was increased to 0.05 cm. The value of $h_h$ is not listed because the same mass hanger was used for five of the seven springs. This hanger had $h_h = 8.000(4)$ cm. The second stiffest spring used a mass hanger with $h_h = 14.806(4)$ cm. These two were measured with a caliper with the error doubled to account for slight awkwardness. The stiffest-spring mass-hanger was measured with a meter stick and had $h_h = 28.2(1)$ cm.

As mentioned in Sec. II B and shown in Fig. 1(b), the other needed lengths can be found from $\ell_2 = h_s + h_h - h_m$ and $\ell_m = h_m/2$. To find an appropriate $h_m$, we estimated a value for the added mass from Eq. (5) and, using standard masses and a vernier caliper, measured $h_m$. If Eq. (42) with this value of $h_m$ predicts a mass with a height that is significantly different than this value of $h_m$, we measured the $h_m$ for the predicted mass and recalculated Eq. (42). One or two iterations were sufficient for consistent results. The last three columns of Table I indicate how much $h_m$ can vary between the predictions from these equations. So, even if we are careful to measure the lengths very precisely, there is an inherent

Table II. Comparison of the predicted mass values to the experimental masses. Except for Eq. (5), the calculation for the predicted mass requires a value for the height of the mass being calculated, $h_m$. As a rough estimate, Eq. (5) is used to estimate $h_m$ in the other equations. To improve the prediction, the result of Eq. (42) is then used to re-estimate $h_m$. This estimation process was self-consistent within three iterations. The two rightmost columns compare the rms average difference $\sigma = \sqrt{\Sigma_{i=1}^{N}(m_i - m_{\mathrm{expt}})^2/N}$ between the predicted and experimental masses. The second-to-last column is averaged over the seven stiffest springs. The last column averages over six springs by excluding the softest and stiffest springs. All units are kilograms. Equations (29) and (42) are significantly better predictors of the mass that experimentally excites the spring-pendulum resonance.

| Eq. used to predict $m$ | Eq. used to estimate $h_m$ | Mass prediction for eight springs, sorted by spring constant, $k$ (N/m) | | | | | | | | rms difference | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 3.059(1) | 6.696(3) | 25.16(4) | 28.63(5) | 36.26(8) | 40.9(1) | 51.4(2) | 1002(2) | all $k$ | $k<100$ |
| (5) | | 0.3052(3) | 0.2765(4) | 0.228(1) | 0.269(1) | 0.334(1) | 0.377(2) | 0.476(3) | 2.29(3) | 3.217 | 0.123 |
| (29) | (5) | −0.1124 | 0.0647 | 0.257 | 0.306 | 0.389 | 0.438 | 0.692 | 9.8 | 0.375 | 0.032 |
| (42) | (5) | −0.1361(6) | 0.070(1) | 0.250(3) | 0.299(4) | 0.381(5) | 0.430(5) | 0.684(7) | 9.2(1) | 0.605 | 0.034 |
| (29) | (42) | −0.1144 | 0.066 | 0.255 | 0.303 | 0.386 | 0.434 | 0.673 | 10.8 | 0.027 | 0.029 |
| (42) | (42) | −0.1339(5) | 0.072(1) | 0.248(3) | 0.296(4) | 0.378(5) | 0.425(5) | 0.663(7) | 10.5(1) | 0.118 | 0.032 |
| (34) | (42) | −0.1460 | 0.051 | 0.185 | 0.231 | 0.302 | 0.354 | 0.488 | 9.64 | 0.449 | 0.102 |
| Experimental range | | N/A | 0.065(5) | 0.245(5) | 0.318(8) | 0.420(10) | 0.480(10) | 0.635(15) | 10.8(2) | | |

systematic uncertainty in that we do not actually know the mass for which we should be measuring the height, $h_m$.

## B. Uncertainty and precision

Table II gives the predictions from the various approximations as well as the experimental values that produce the resonance. The experimental masses were found by trial and error starting from the predicted values. The central value is that which subjectively seemed to give the most pronounced effect. The error bars on the experimental mass are the extent to which I could adjust the mass and see a pendular swing that retained a small amount of bouncing. Including masses that produced a moderate swing without significantly diminishing the vertical oscillation would typically double or triple the listed uncertainty.

Table II also estimates the precision of some predicted values. Because Eq. (5) is a product, the relative uncertainties can be added in quadrature:

$$\sigma_{m,5} = \frac{k\ell_0}{3g}\sqrt{\left(\frac{\sigma_k}{k}\right)^2 + \left(\frac{\sigma_\ell}{\ell_0}\right)^2 + \left(\frac{\sigma_g}{g}\right)^2}. \tag{43}$$

The precision of Eqs. (29) and (34) could be found by a Monte Carlo analysis, but was not due to the consistency between Eqs. (29) and (42) and the measurements. The determination of the precision of Eq. (42) is only slightly more complicated than Eq. (43) due to the additional terms:

$$\sigma_{m,42} = \frac{k(\ell_0 + 3\ell_{\mathrm{ave}})}{3g}$$
$$\times \sqrt{\left(\frac{\sigma_k}{k}\right)^2 + \left(\frac{4\sigma_\ell}{\ell_0 + 3\ell_{\mathrm{ave}}}\right)^2 + \left(\frac{\sigma_g}{g}\right)^2} + \frac{2}{3}\sigma_{\mathrm{ms}}. \tag{44}$$

The relative uncertainty in $4\sigma_\ell/(\ell_0 + 3\ell_{\mathrm{ave}})$ dominates the right-hand side of Eq. (44). The relative uncertainty of $k$ and $g$ are below 1% and usually much less. The relative uncertainties $\sigma_\ell/\ell_0$ and $4\sigma_\ell/(\ell_0 + 3\ell_{\mathrm{ave}})$ are such that for every millimeter that $\sigma_\ell$ is increased, $\sigma_m/m$ increases by about 1%. This important point cannot be emphasized enough to those who wish to utilize these equations. If the lengths are not measured carefully, then the precision of the predictions is reduced. Further, we must control the systematic uncertainty of using a precise measurement of an inaccurate pre-

diction. This can be done by checking the self-consistency mentioned at the end of Sec. IV A.

To be explicit, the $4\sigma_\ell$ term appears in Eq. (44) as the uncertainty in $\ell_s + \ell_1 + \ell_2 + \ell_m$; however, these uncertainties are not actually all the same. Some terms are measured with a meter stick (precision of 0.05 cm) and some with a caliper (precision of 0.002 cm). For awkward measurements, these precisions were increased. Further, $\ell_2 = h_s + h_h - h_m$ and each of these measurements increases the uncertainty. With this in mind, $4\sigma_\ell$ is actually 0.1 cm (from $\ell_s$) + 0.05 cm (from $\ell_1$) 0.05 cm (from $h_s$ in $\ell_2$) + 0.004 cm (from $h_h$ in $\ell_2$) + 0.004 cm (from $h_m$ in $\ell_2$) + 0.002 cm (from $\ell_m = h_m/2$) = 0.210 cm. If the added mass or the hanger is measured with a ruler (as for the stiffer springs that require more applied mass), $4\sigma_\ell$ becomes 0.256 cm.

Regarding the systematic uncertainty, we see from Table I that $h_m$ can be off by as much as 2 cm if the mass is not predicted accurately. If $\sigma_\ell$ is taken to be 0.5 cm so that the $4\sigma_l$ term matches this 2 cm uncertainty, then $\sigma_m/m \approx 5\%$. For the 51 N/m spring which requires an added mass of 0.635 kg, this 5% relative uncertainty, $\sigma_m/m$, implies an uncertainty of $\sigma_m = 0.032$ kg. Unfortunately, Table II shows that I only obtained a good resonance within 0.015 kg of 0.635 kg. This variation indicates that it is important to minimize the systematic uncertainty by verifying the consistency between the mass used to estimate $h_m$ and the predicted mass. However, this may not be of dire necessity when one considers that with $\sigma_\ell \approx 0.5$ cm, the prediction of Eq. (29) is still within $2\sigma_m$ of the experimental value.

The uncertainties listed for the masses in Table II reflect the roughly 1% precision of the length measurements. That these uncertainties do not overlap with the experimental values, especially using Eq. (29), suggests that there is an additional source of uncertainty. This discrepancy will be considered in Sec. IV C where we will discuss the underlying assumptions.

The precision quoted for the predictions in Table II indicates only how much the result will vary due to the measurement uncertainties. It does not indicate how close the prediction is to the experimental value, because it does not include the systematic uncertainties such as the 2 cm variation in the prediction of $h_m$. In addition, no attempt was made to indicate the size of the terms dropped in the approximations

because doing so would only indicate how far the various approximations are from Eq. (29), not how far the prediction is from the experimental value.

To estimate the quality of the predictive value of each expression, the rightmost columns of Table II show the root-mean-square difference between each predicted mass and the experimental mass averaged over the various springs:

$$\sigma = \sqrt{\frac{1}{N}\sum_{\mathrm{eq}=1}^{N}(m_{\mathrm{eq}}-m_{\mathrm{expt}})^2}. \tag{45}$$

Two values are given for each equation. The second-to-last column is the rms error including all springs. The last column is the rms error not including the stiffest spring. Because the stiffest spring involves masses that are over an order of magnitude larger, these differences can significantly overwhelm the others. On average, Eq. (29) predicts the experimental value to within 0.027 kg, even when the stiffest spring is included.

## C. The results

As expected, Table II shows that Eqs. (29), (34), and (42) are significant improvements over the assumption of a massless spring in Eq. (5). For the smallest $k$ spring in Table II, the negative mass values are consistent with not finding an experimental value. Although the prediction of Eq. (5) for this spring is 0.305 kg, the measured period of swinging and of bouncing indicates that $T_p$ cannot equal $2T_s$ as needed for the resonance. In fact, the massless spring approximation, Eq. (5), only has a chance of being coincidentally correct for $k \approx 20$ N/m. For $k$ above this value, Eq. (5) predicts a mass that is too small and becomes worse for larger values of $k$, in spite of the expectation that this approximation might improve for stiff springs. In fact, as $k$ increases, the need to include $\ell_1$, $\ell_2$, and $\ell_m$ becomes more relevant, presumably because a stiffer spring requires more mass to oscillate (making at least $\ell_m$ harder to ignore) and the inclusion of these terms is a better measure of the pendular length, which controls the pendular period. Because the long-spring approximation, Eq. (31) ($\ell_1 = \ell_2 = \ell_m = 0$), is necessarily smaller than the massless spring of Eq. (5), the predictions of Eq. (31) were not included in Table II.

As seen in the rms averages of Table II, the consistency between Eqs. (42) and (29) is impressive. Although the rms difference over all springs is 0.605 kg for Eq. (42), it predicts the mass to within 3% for the stiffest spring and has an rms difference of only 0.032 kg for the other springs. This predictive capability is not too different from Eq. (29), which has an rms difference below 0.03 kg whether the stiff spring is included or not. If Eq. (42) does not give accurate results and one wishes to avoid solving a cubic polynomial, one could use the calculated mass to estimate the $\mathcal{O}(y)$ terms in Eq. (40) to determine how far this prediction is from Eq. (29).

It is interesting that Eq. (29) does not give exact results. It is possible that the problem is due to the systematic uncertainty discussed in Sec. IV B. As noted there, for half of the springs, the predictions of Eq. (29) are within $\sigma_m$ of the experimental value and within $2\sigma_m$ for the remaining springs if we assume that $\sigma_\ell = 0.5$ cm. This is a sizable uncertainty for the measurement of lengths and so this estimate may only account for a portion of the difference. Another explanation for the inaccuracy of Eq. (29) might come from the assump-

tions that the spring is a rigid rod and has uniform density. Equation (42) uses only these assumptions and the inclusion of $\ell_1$, $\ell_2$, and $\ell_m$ and gives predictions that are comparable to Eq. (29) for the five central springs. Equation (42) is within about 3% of the experimental value for the very stiff spring, where one might expect these assumptions to be more reasonable, and is about 11% off for the softest spring, where these assumptions are more suspect. Equation (29) is close in both cases. As mentioned, the rigid rod assumption should be reasonable because the swing mode at the resonance mass does not have a discernible bounce. The uniform density assumption, on the other hand, might warrant further investigation. See Appendix B for further details.

## V. CONCLUSIONS

The purpose of this paper is to improve the prediction for the mass, not to solve the differential equations of motion that would generalize the analysis of Refs. 1 and 2 describing the motion of the oscillating mass. Although the expression that predicts the mass for a massless spring as a simple pendulum, Eq. (5), is easy to derive, it is possible to improve the prediction by extending the derivation to describe the massive spring as a physical pendulum. This extension gives Eq. (29), a cubic polynomial for the mass that will induce the spring-pendulum resonance. A rough approximation also was developed that gives a much simpler expression, Eq. (42), and comparable results. It is clear from Table II that both Eqs. (42) and (29) are much better predictors of the mass than Eq. (5). Equations (29) and (42) assume that the swinging spring is a rigid rod with uniform density. Because in full swing, it does not bounce, the rigid rod assumption is probably reasonable, at least for springs that are not too long. However, the uniform mass assumption is somewhat suspect (see Appendix B). The lengths $\ell_2$ and $\ell_m$ (shown in Fig. 1) also play a significant role in the accuracy of the predictions. The precision of the predictions in Table II are due primarily to the uncertainty in the lengths.

## APPENDIX A: DETAILS OF MEASURING $k/g$

Eighteen masses were gradually added to each spring and the corresponding stretch was measured using a mirrored ruler to minimize parallax. The masses used to cause the stretch ranged from 0.150 to 1 kg for the springs with mid-range stiffness, 0.010 to 0.500 kg for the soft spring, and 3 to 15 kg for the stiff spring. The first two mass ranges were measured to a precision of 0.1 g; the third was measured to within 2 g. The corresponding displacement was measured to 0.1 cm for all but the two softest springs. Due to limitations in construction, these were measured to a precision of 0.2 cm. The slope was found by a least-squares fit[16] to account for the measurement uncertainty in both the added mass and the stretch. To give an idea of the value of the spring constants for the springs used, we assume $g = 9.80(1)$ m/s$^2$ and divide by the slope $g/k$. This value is listed as $k$ in Tables I–III. The reduced $\chi^2$ of these fits, $\chi^2_\nu = \chi^2/\nu$ with $\nu$ equal to the degrees of freedom, are listed in Table III. Each spring has eighteen data points except for the $k = 25$ N/m spring. The value of $\chi^2_\nu \approx 0.3$ indicates that the precision in the displacement may have been overestimated, so the $k$ values

Table III. Tests of the uniform stretch (coil density) assumption. The values of $k$ are from the slope of a graph of $m$ vs $x$, assuming that $g=9.80(1)$ m/s$^2$. (The reduced chi-square, $\chi_\nu^2$, values indicate that the precision of the displacement may be overestimated.) The ratio $Z_{\text{c.m.}}/\ell$ compares the distance from the top of the spring to the calculated center of mass, $Z_{\text{c.m.}}$, to the length of the spring, $\ell$. This ratio is evaluated at the experimental mass value and deviations from 50% indicate the nonuniformity of the stretch. A graph of $(T/2\pi)^2$ vs mass has slope $s$ and intercept $b$. If we assume uniform stretch as discussed in Appendix B, the reciprocal slope $(1/s)$ should give $k$ (without assuming a value of $g$), the ratio $3b/s$ should give the mass of the spring, and the ratio of $b/(sm_s)$ should be 1/3. Differences from these values should correlate to differences in $Z_{\text{c.m.}}/\ell$ from 50%.

| $k$ (N/m) | $\chi_\nu^2$ | $\nu$ | $Z_{\text{c.m.}}/\ell$ (%) | $m_s$ (kg) | $1/s$ | $3b/s$ | $b/(sm_s)$ |
|---|---|---|---|---|---|---|---|
| 3.059(1) | 1.196 | 16 | 55.08 | 0.6779(2) | 3.260(8) | 0.850(3) | 0.418(2) |
| 6.696(3) | 0.336 | 16 | 52.67 | 0.3390(2) | 6.837(9) | 0.359(1) | 0.353(1) |
| 25.16(4) | 0.309 | 15 | 50.71 | 0.0829(2) | 25.44(6) | 0.098(4) | 0.40(2) |
| 28.63(5) | 0.355 | 16 | 50.60 | 0.0824(2) | 28.81(5) | 0.092(3) | 0.37(1) |
| 36.26(8) | 0.316 | 16 | 50.51 | 0.0884(2) | 36.56(6) | 0.095(3) | 0.36(1) |
| 40.9(1) | 0.379 | 16 | 50.44 | 0.0846(2) | 41.14(5) | 0.094(3) | 0.37(1) |
| 51.4(2) | 0.164 | 16 | 50.31 | 0.0786(2) | 51.9(1) | 0.085(4) | 0.36(2) |
| 1002(2) | 0.494 | 16 | 50.03 | 0.0629(2) | 1015(3) | 0.07(9) | 0.4(5) |

assume the common uncertainty: $\sigma_c^2 = \sigma_{i,\text{ave}}^2 \chi_\nu^2$, which gives $\sigma_c = 0.03 - 0.06$ cm for all but the two softest cases, which have $\sigma_c = 0.3$ cm.

## APPENDIX B: NONUNIFORM COIL DENSITY IN A MASSIVE SPRING

In a massive spring, each coil stretches all of the coils above it and none of the coils below it, which implies a nonuniform distribution of the coils of the spring. When hung under its own weight, the density must increase toward the bottom. The mass per coil is still considered uniform; it is the nonuniform distribution of the coils that produces a nonuniform distribution of the spring mass. In fact, the mass density is

$$\lambda(z) = \lambda_0 \frac{m_s}{\ell_0 + \dfrac{mg}{k} + \dfrac{\ell - z}{\ell}\dfrac{m_s g}{k}}, \tag{B1}$$

where $\lambda_0$ is a dimensionless normalization constant given by

$$\lambda_0 = \frac{m_s g}{k\ell}\left[ \ln\left( \frac{\ell_0 + \dfrac{mg}{k} + \dfrac{m_s g}{k}}{\ell_0 + \dfrac{mg}{k}} \right) \right]^{-1} \tag{B2}$$

and $\ell = \ell_0 + (g/k)(m + m_s/2)$. Because the argument of the logarithm also can be expressed as $(\ell + m_s g/2k)/(\ell - m_s g/2k)$, it is possible to write $\lambda_0$ as

$$\lambda_0 = \frac{m_s g}{k\ell}\left[ \tanh^{-1}\left( \frac{m_s g}{2k\ell} \right) \right]^{-1}. \tag{B3}$$

We can verify that $\int_0^\ell \lambda(z)dz = m_s$, that the average density matches the uniform density used in Sec. II A, and that

$$Z_{\text{c.m.}} \equiv \frac{1}{m_s}\int_0^\ell z\lambda(z)dz, \tag{B4}$$

$$Z_{\text{c.m.}} = \ell\left[ \frac{\ell_0 + \dfrac{(m+m_s)g}{k}}{\dfrac{m_s g}{k}} - \frac{1}{\ln\left( \dfrac{\ell_0 + (m+m_s)g/k}{\ell_0 + mg/k} \right)} \right], \tag{B5}$$

$$Z_{\text{c.m.}} \overset{m_s/k \to 0}{\longrightarrow} \frac{\ell}{2} - \frac{1}{12}\frac{m_s g/k}{\ell_0 + mg/k} + \mathcal{O}\left( \frac{m_s g}{k} \right)^2. \tag{B6}$$

Equation (B5) shows that Eqs. (22) and (27) should be modified significantly to account for the shift in the center of mass. Equation (B6) shows that for either a light spring or a stiff spring, the center of mass approaches the midway point, $\ell/2$.

As an initial test of the nonuniformity of the springs, Eq. (B6) was used to calculate the location of the center of mass for the static hanging spring. If the spring is perfectly uniform in stretch, then $Z_{\text{c.m.}}$ should be 50% of $\ell = \ell_0 + mg/k + m_s g/2k$. Table III lists the ratio $Z_{\text{c.m.}}/\ell$ for the springs used in this experiment. The mass used was that predicted by Eq. (29). The value for the ratio will, of course, change with different $m$. The assumption of uniform density might warrant further investigation for better accuracy in the softer springs, but from $Z_{\text{c.m.}}$, the assumption seems reasonable for all but the softest springs.

After relaxing the uniform density assumption, Eq. (10) can still be integrated, but is rather more involved. It is possible to derive a (fairly complicated) formula for the effect of the spring mass in terms of $\ell_0$, $k$, $m$, and $m_s$, which does not seem more useful than previous approximations.

On the other hand, it is possible to account for variations in the $m_s$ coefficient without doing any more work than it would take to measure the spring constant, which can be done using Eq. (14) rather than via Hooke's law. First attach the spring to the desired support, which ideally will not flex with the bouncing of the spring. Then, for a collection of masses, let the spring oscillate and measure each period. If we rearrange Eq. (14) as

$$\left( \frac{T}{2\pi} \right)^2 = \frac{1}{k}m + \frac{m_s}{3k}, \tag{B7}$$

we see that a plot of $(T/2\pi)^2$ vs $m$ will give a straight line with slope $1/k$ and an intercept that, when divided by the slope, should be close to $m_s/3$. Any deviation from $m_s/3$ would indicate the value that should be used in Eq. (14). We can use the ratio of intercept divided by the slope and the actual mass of the spring in place of the 1/3 that currently multiplies $m_s$ in Eq. (14). In other words, the ratio of intercept divided by the slope is a direct measurement of the $D$ that appears in Eq. (16).

As a second test of the nonuniform density, the spring constant for each spring was found by measuring the period of 18 different masses over 100 oscillations. (Fifty oscillations were used for the soft springs.) The graphs of $(T/2\pi)^2$ vs $m$ are available from the author. The last three columns of Table III are from this graph. The reciprocal of the slope $1/s$ should give the spring constant. These differ from those listed in Table I, because Table I assumes $g=9.80(1)$ m/s$^2$. Three times the intercept divided by the slope $(3b/s)$ should be close to $m_s$. Equivalently, the quantity $b/(sm_s)$ should be close to 1/3, the coefficient used for $m_s$. These last two comparisons are equivalent measures of the accuracy of the assumption of uniform coil density. The value of $b/(sm_s)$ for the stiffest spring also is consistent with 1/3, having a somewhat imprecise result of 0.4(5). Interestingly, the softest spring has a larger ratio, 0.418(2), which is inconsistent with the $4/\pi^2 = 0.405\,284\,7$ predicted by Refs. 6, 8, 9, and 14. Except for the stiffest and softest springs, the weighted average of the results is 0.354(1), which indicates a difference from 1/3 and might explain why the cubic error bars do not overlap the experiment in every case.

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[5]Equation (25) in Ref. 2 for the period with which the energy oscillates between these modes involves elliptical integrals of the first kind, even for the case of a massless spring as a simple pendulum.

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[9]J. T. Cushing, "The spring-mass system revisited," Am. J. Phys. **52** (10), 925–937 (1984). See footnote 16.

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[11]R. J. Stephenson, *Mechanics and Properties of Matter* (Wiley, New York, 1952), pp. 113–114. See also L. Ruby, "Equivalent mass of a coil spring," Phys. Teach. **38**, 140–141 (2000) for the same result from iterative methods to avoid the calculus for an algebra-based course. Ruby also attempts a simplified nonuniform approximation, but incorrectly relates Mak's (Ref. 7) static correction to his dynamic correction.

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[15]As mentioned in this article, a mass hanging in equilibrium from a spring will satisfy $kz=mg$. A graph of displacement versus added-mass then gives a slope of $g/k$. As described in Appendix B, a mass that is hung from a spring and set to oscillating has a characteristic period. A plot of $[T/2\pi]^2$ vs $m$ then gives a line with slope $1/k$. In principle, $g/k$ divided by $1/k$ will give a calculation of the local gravitational field. In a paper in preparation, these values are compared to the acceleration due to gravity measured by a spark-machine free-fall experiment and the gravitational field measured by a swinging pendulum. The free-fall experiment gave repeatable results consistently lower than expected. The springs and pendula independently gave repeatable results that were higher than expected and consistent with each other. The systematic uncertainties are still being investigated.

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