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Derivation of expression for Young's modulus

Let us consider a beam initially unstressed as shown in Fig 1(a). Now the beam is subjected to a constant bending moment (i.e. 'Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b).

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis. The radius of curvature R is then measured to this axis.

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF**, originally parallel as shown in fig 1(a). when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'**, the final position of the sections, are still straight lines, they then subtend some angle 'θ'.

Consider now fiber AB in the material, at a distance 'z' from the neutral surface (N.A.), when the beam bends this will stretch to A'B'

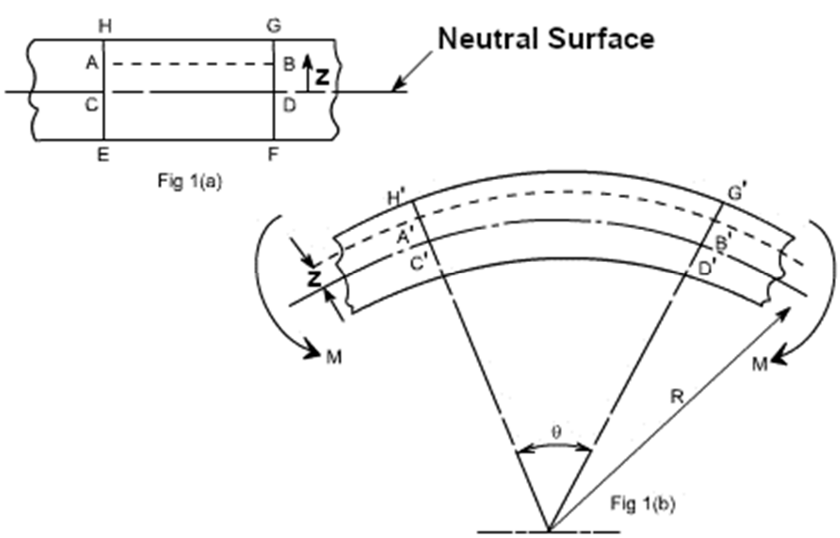

Fig. 1

Therefore, strain in fiber AB = $\frac{change\ in\ length}{original\ length} = \frac{A'B' - AB}{AB}$. But from the figures 1(a) and 1(b), AB = CD and CD = C'D'

Therefore, strain(s) = $\frac{A'B'-C'D'}{C'D'} = \frac{(R+z)\theta-R\theta}{R\theta} = \frac{z}{R}$.

However, $\frac{stress\ (\sigma)}{strain\ (s)} = Youngs\ modulus\ (E)$.

Therefore $\frac{\sigma}{E} = \frac{z}{R}$ (A1)

Consider any arbitrary cross section of beam as shown in Fig.2. Strain on a fiber at a distance 'y', from the neutral axis (N.A.), is given by the expression $\sigma = \frac{E}{R}y$

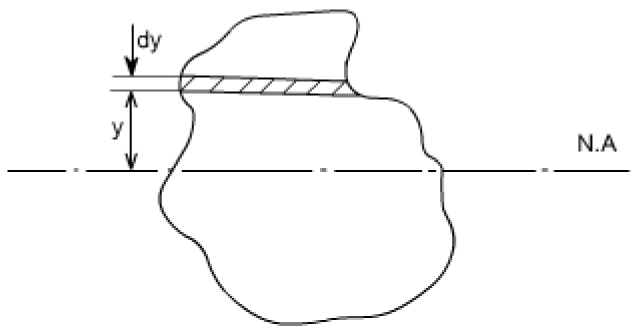

Fig.2

If the shaded strip is of area 'dA' then the force on the strip is $F = \sigma\Delta A = \frac{E}{R}y\delta A$

Bending moment (or the moment of resistance to bending) about the neutral axis = F·y = $\frac{E}{R}y^2\ \delta A$

The total moment for the whole cross-section is therefore equal to $M = \sum\frac{E}{R}y^2\ \delta A = \frac{E}{R}\sum y^2\delta A$

Now the term $\sum y^2\ \delta A$ is the property of the material and is called geometrical moment of inertia (I_g) or second moment of inertia.

Therefore, bending moment $M = \frac{E}{R}I_g$ (A2)

Bending of the rod with a load at the centre is shown in Fig. 3. Consider right hand sided half portion of the rod described with solid lines. It is equivalent to a cantilever with a fixed end at O.

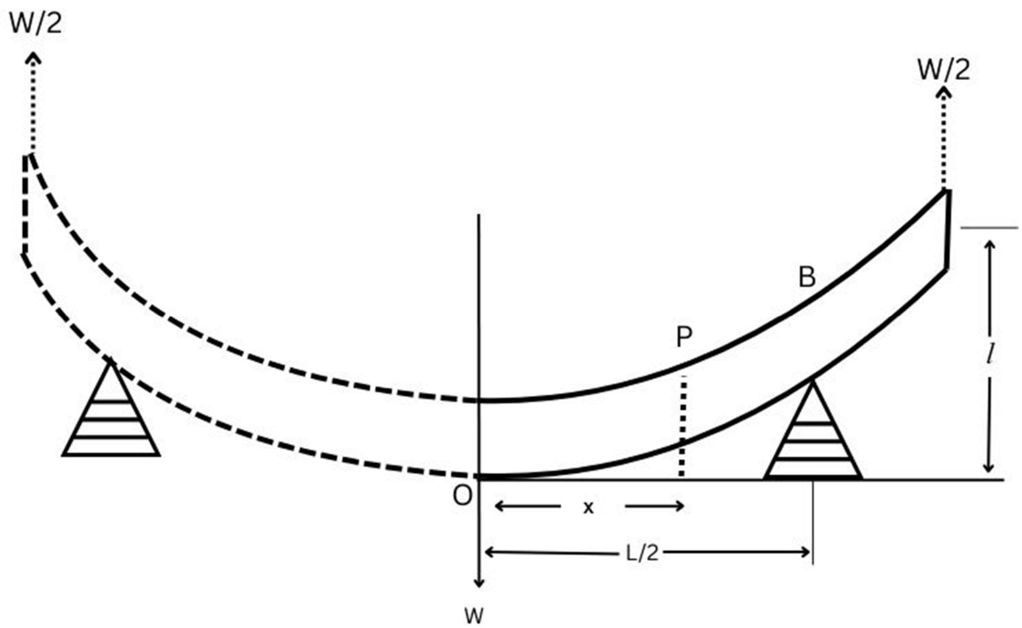

Fig.3

Considering the section PB of the cantilever OB, at a distance 'x' away from its fixed end O,

Moment of bending couple due to load of weight '$W/_2$' is equal to $\frac{W}{2}\left(\frac{L}{2}-x\right)$ (A3)

The rod being in equilibrium, this must be balanced by bending moment (or the moment of the resistance due to bending)

From (A2) and (A3) $\frac{W}{2}\left(\frac{L}{2}-x\right)=\frac{E}{R}I_g$ (A4)

The radius of curvature can be mathematically expressed in terms of derivative ($\frac{dl}{dx}$) and second derivative $\left(\frac{d^2l}{dx^2}\right)$ as $R=\frac{\left(1+\left(\frac{dl}{dx}\right)^2\right)^{3/2}}{\left(\frac{d^2l}{dx^2}\right)}$

For small curvatures, ($\frac{dl}{dx}$) is small and is $(\frac{dl}{dx})^2$ too small compared to 1.

Therefore $\frac{1}{R}=\frac{d^2l}{dx^2}$

By substituting for R and integrating, $\frac{dl}{dx}=\frac{W}{2EI_g}\left(L\frac{x}{2}-\frac{x^2}{2}\right)+C$ (A5)

At the end of the cantilever (O), there is no bending, meaning at x = 0 on the cantilever dl/dx = 0.

Therefore, it implies C = 0, substituting C = 0 in (A5)

and on further integrating between the limits x= 0, and x = L/2 gives

$$l = \frac{WL^3}{48\, EI_g} \qquad \text{(A6)}$$

For a rectangular cross section with breadth *b* and depth *d*, geometrical moment of inertia

$I_g = \frac{bd^3}{12}$ (how? clue: For a rectangular cross section bar $I_g = \int y^2\, dA$)

By substituting for I_g in A6, $l = \frac{WL^3}{4\, E\, b\, d^3}$ or $E = \frac{WL^3}{4\, l\, b\, d^3}$

References:

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3) Introduction to Structural Mechanics by P.S.Smith- Palgrave McMillan (2007)
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