A useful superconductor

Peter Finch
James De Lisle
Gian Palumbo
JKP

PRL 2015

EPSRC
Engineering and Physical Sciences
Research Council

UNIVERSITY OF LEEDS

CTCMP-2015, NISER Bhubaneswar

Motivation

- **Quantum Computation** main challenges:
  - finite temperature
  - experimental imperfections
- Employ **Topological Quantum Systems:**
  - Interacting: **Fractional Quantum Hall Effect**
  - Non-interacting: **Superconductivity (TSC)**

Majorana fermions

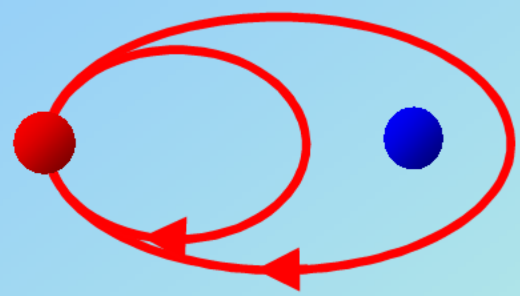
Anyons

[Wilczek, Freedman, Wen, Bais, Wang, Kitaev,...]

Anyons

- Dynamically trivial (H=0), but *hard core.*
- Only **statistics:**

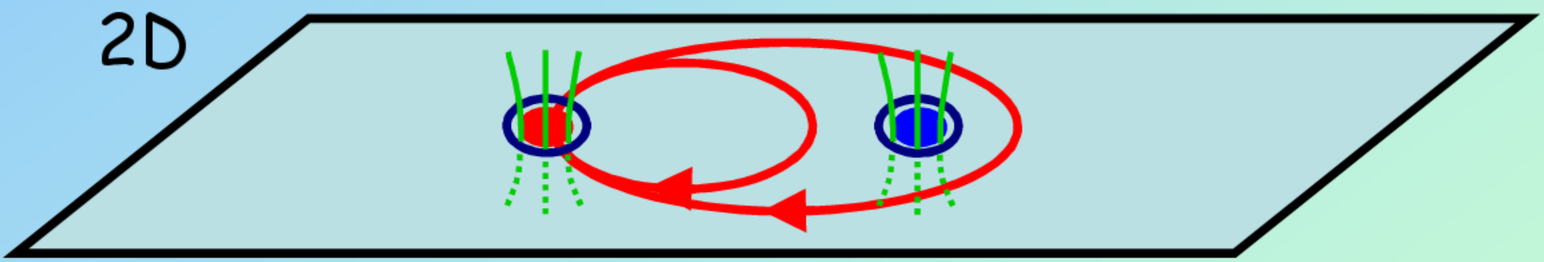
3D



Bosons $|\Psi\rangle \to |\Psi\rangle$

Fermions $|\Psi\rangle \to e^{i2\pi}|\Psi\rangle$

2D



$|\Psi\rangle \to e^{i2\varphi}|\Psi\rangle$

$|\Psi\rangle \to U|\Psi\rangle$

Anyons

Anyons: vortices with **flux** & **charge (fractional)**.

Aharonov-Bohm effect **Berry Phase**.

Introduction

Majoranas are met in:

- **High energy physics** (3+1) dims:

  elementary "real" fermions
- **Condensed matter** (2+1) dims:

  quasiparticles, non-Abelian **anyons**
- **Alternative** representation of superconductors:
  - 1 dim quantum wires (fermions, Ising spin-1/2)

    [see also K. Sengupta talk next]
  - 2 dims p-wave superconductors

    [see also G. Baskaran talk next]
  - 3 dims topological systems

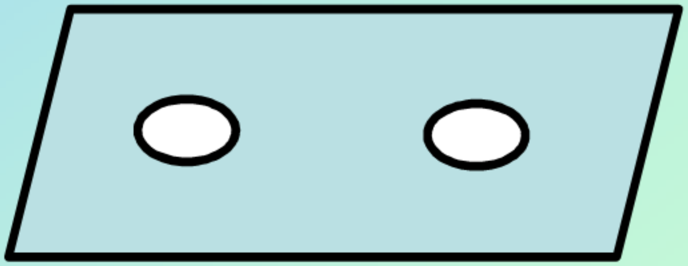
Introduction

Majoranas:

Fermions that **are their own anti-particles**

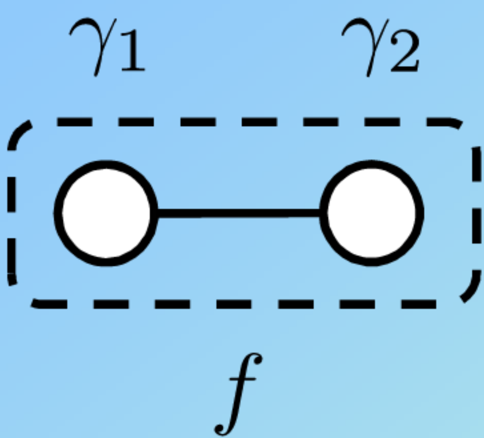
$$\gamma\gamma^\dagger + \gamma^\dagger\gamma = 2 \qquad \gamma^\dagger = \gamma \qquad \gamma^2 = 1$$

When they are restricted to 2 dims they **also behave as non-Abelian anyons**



Majoranas: 0 Dim

2 Majoranas = 1 normal fermion



$$f = \frac{\gamma_1 + i\gamma_2}{2} \qquad f^\dagger = \frac{\gamma_1 - i\gamma_2}{2}$$

$$f f^\dagger + f^\dagger f = 1$$

Fermionic mode occupation $\quad f^\dagger f = 0, 1$

$$f^+|0\rangle = |1\rangle,\ f|1\rangle = |0\rangle,\ f|0\rangle = 0,\ f^+|1\rangle = 0$$

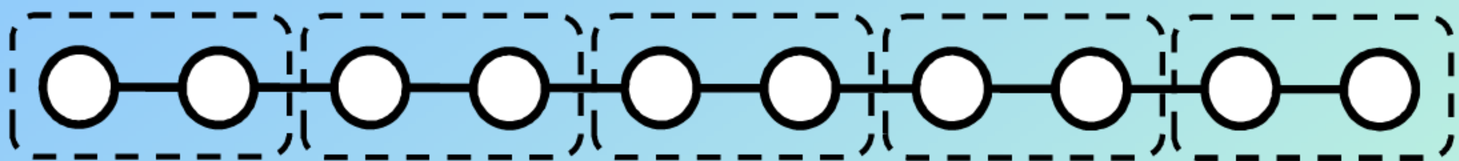
Introduce Hamiltonian:

$$H = -\mu(f^+ f - \frac{1}{2}) = -\frac{i\mu}{2}\gamma_1\gamma_2$$

Majoranas: 1 Dim

Even number of Majoranas: quantum wire

$\gamma_{i,1}\ \gamma_{i,2}$



f_i

$$f_i = \frac{\gamma_{i,1} + i\gamma_{i,2}}{2} \qquad f_i^\dagger = \frac{\gamma_{i,1} - i\gamma_{i,2}}{2}$$

Hamiltonian of superconductor: $H = \sum_{k=1}^{2L} t_k \gamma_k \gamma_{k+1}$

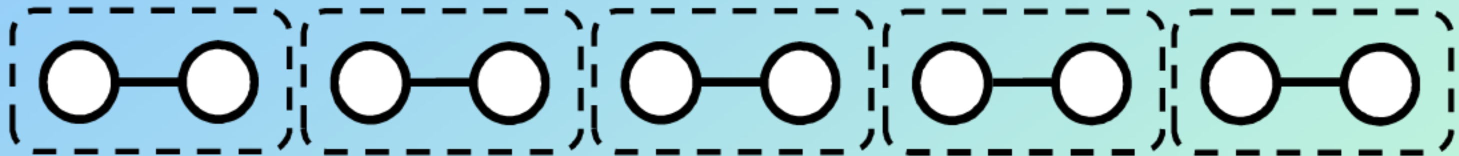
$$H = \sum_{j=1}^{L} \left[-w \left(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j \right) - \mu \left(f_j^\dagger f_j - 1/2 \right) + \left(\Delta f_j f_{j+1} + \Delta^* f_{j+1}^\dagger f_j^\dagger \right) \right]$$

[see also K. Sengupta talk next]

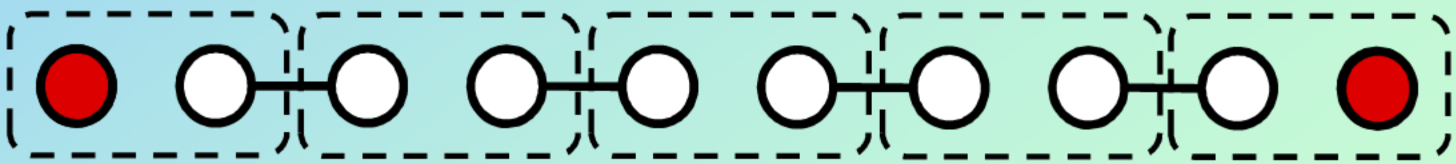
Majoranas: 1 Dim

- If $w = \Delta = 0$

$$H = -\frac{i\mu}{2}\sum_{i=1}^{L}\gamma_{i,1}\gamma_{i,2} = -\mu\sum_{i=1}^{L}(f_i^\dagger f_i - \frac{1}{2})$$



- If $w = \Delta,\ \mu = 0$ $\qquad H = iw\sum_{i=1}^{L-1}\gamma_{i,2}\gamma_{i+1,1}$



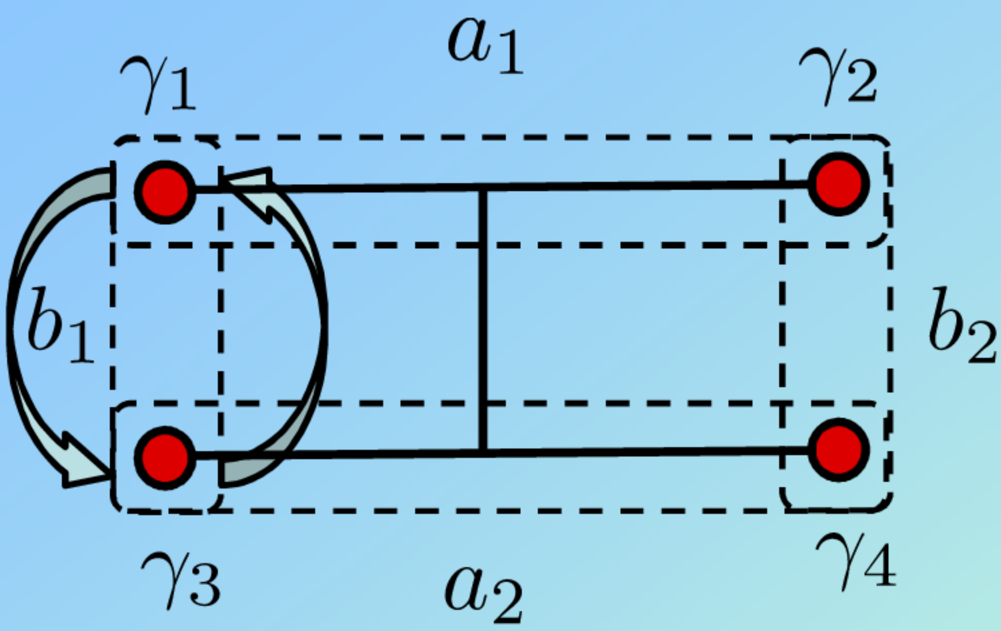
Majoranas appear at the edge of the wire:

localised quasiparticles

Zero energy E=0

Majoranas: 1 Dim

Anyonic properties of Majorana fermions:



$$\{a_1, b_1\} = \frac{1}{2}$$

Fusion

$$|11\rangle_a = \frac{1}{\sqrt{2}}(|00\rangle_b + |11\rangle_b)$$

$$|00\rangle_a = \frac{1}{\sqrt{2}}(|00\rangle_b - |11\rangle_b)$$

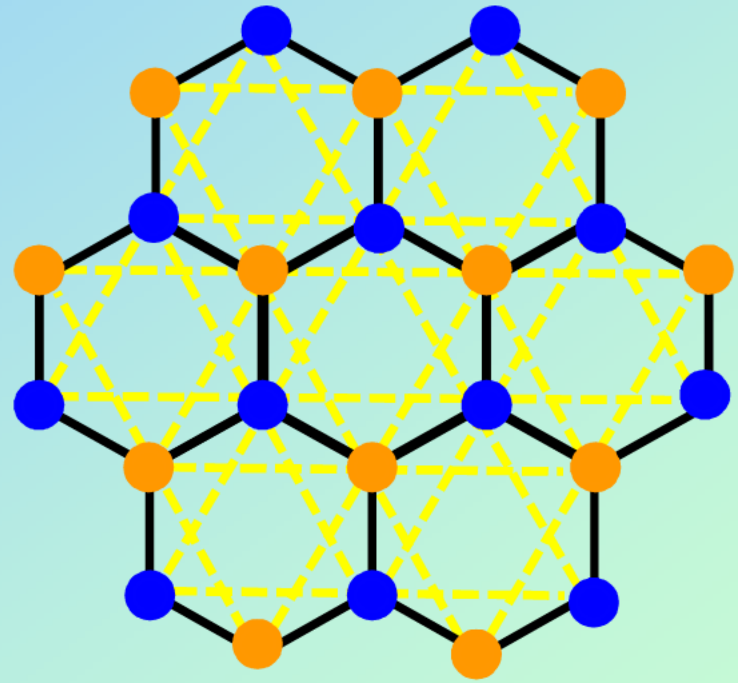
Braiding

$$\mathcal{U} = a\mathbf{1} + b\gamma_1 + c\gamma_3 + d\gamma_1\gamma_3$$

$$\mathcal{U}^2 = e^{i\pi/4}\gamma_1\gamma_3 = e^{i\pi/4}(a_1 a_2 + a_1 a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)$$

Majoranas: 2 Dim

Kitaev's honeycomb lattice



$$H = \pm i \sum_{\langle i,j \rangle} \gamma_i \gamma_j$$

- Analytically tractable:

2D TSC of **type D** (PH symm./no TR-symm.)

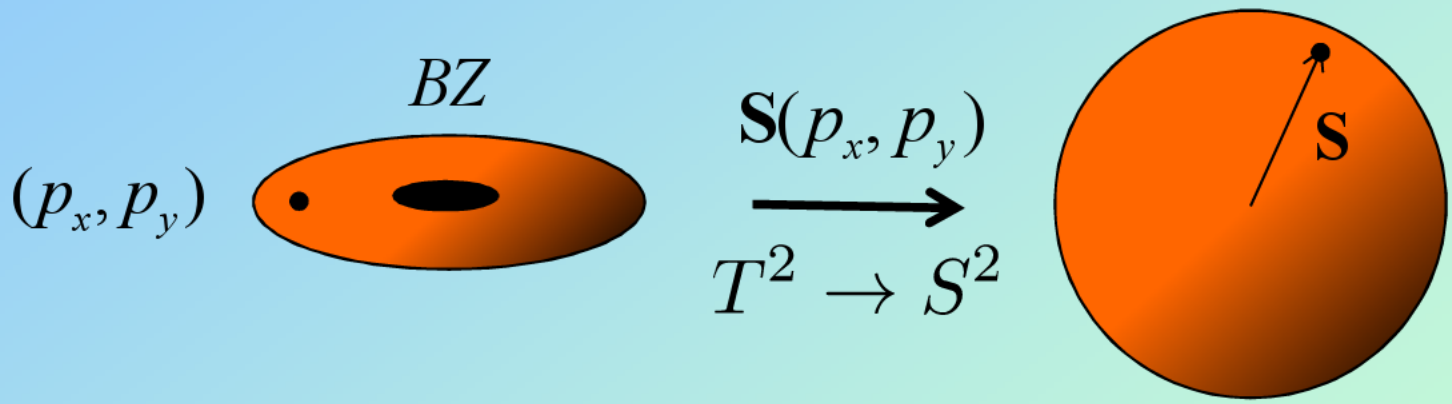
- It supports **vortices** that behave like **Majorana fermions** (same as in 1 Dim)

[see also G. Baskaran talk next]

Majoranas: 2 Dim

Kitaev's honeycomb lattice

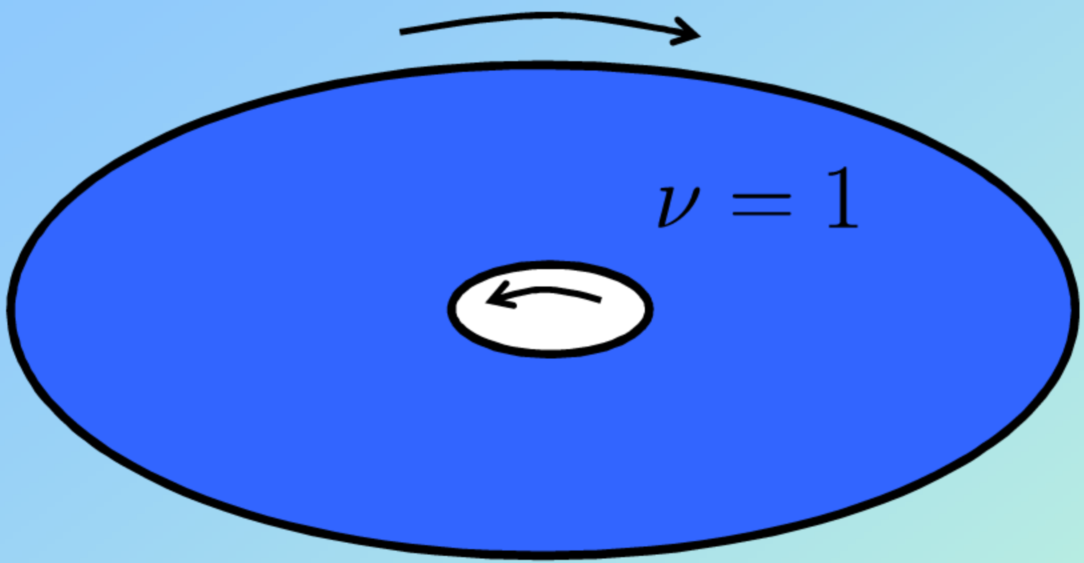
$$H = \sum_{\mathbf{p}} (f^\dagger_{\mathbf{p}}\, f_{-\mathbf{p}}) h(\mathbf{p}) \begin{pmatrix} f_{\mathbf{p}} \\ f^\dagger_{-\mathbf{p}} \end{pmatrix} \qquad h(\mathbf{p}) = E(\mathbf{p})\mathbf{S}(\mathbf{p}) \cdot \boldsymbol{\sigma}$$

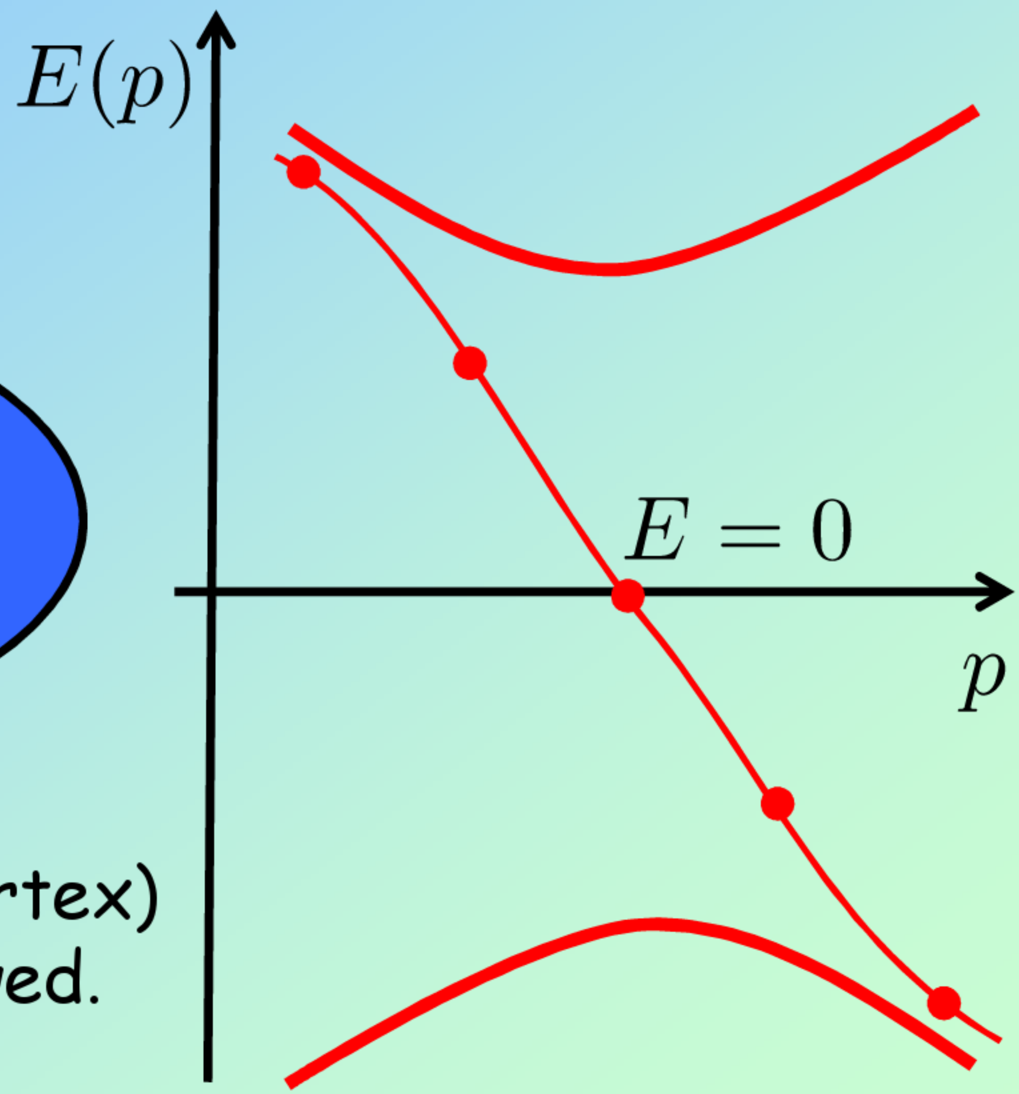


- **Chern number:** $\nu = \frac{1}{4\pi} \int_{BZ} \mathbf{S} \cdot \left(\partial_{p_x} \mathbf{S} \times \partial_{p_y} \mathbf{S} \right) d^2 p$
- **Particle-hole symmetry:** $\Psi^\dagger_E \longleftrightarrow \Psi_{-E}$

Majoranas: 2 Dim

Kitaev's honeycomb lattice





If π-flux through the hole (vortex) then zero energy mode is allowed.

PH-symmetry -> $\Psi^{+}_{E=0} = \Psi_{-E=0}$: single Majorana fermion.

Majoranas: 3 Dim

Questions:

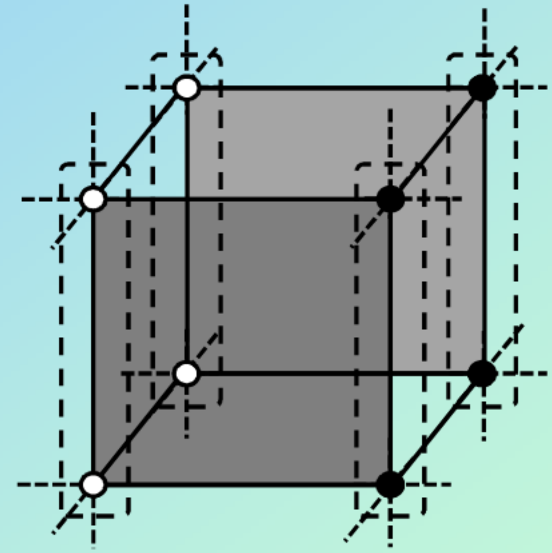
- Can we build **3D TSC** -- out of interacting Majoranas?
- What are the edge (surface) states?

Bonus:

- Unexpected **robust 2D topological phase** emerges.

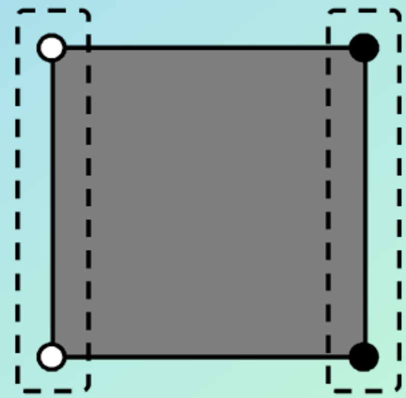
PRL 114, 016801 (2015)

Majoranas: 3 Dim



Cubic lattice:

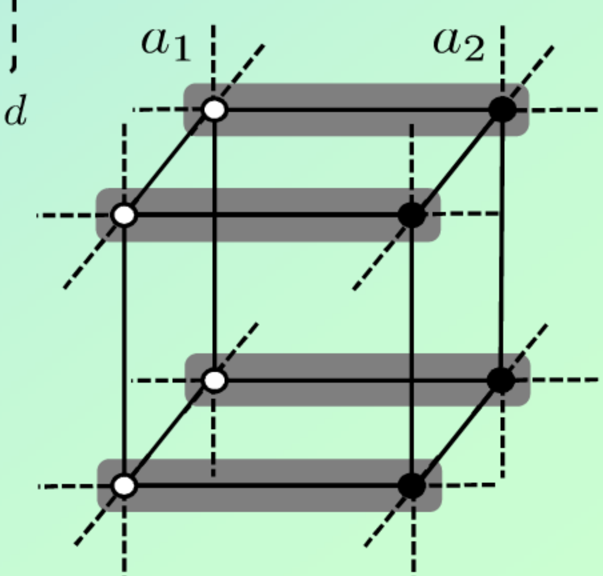
$$H = \pm i \sum_{\langle i,j \rangle} \gamma_i \gamma_j \qquad |i - j| \leq \sqrt{5}$$



Fermions: $k = 1, 2$

$$a_k = \frac{\gamma_{k,d} + i\gamma_{k,u}}{2}$$

$$a_k^\dagger = \frac{\gamma_{k,d} - i\gamma_{k,u}}{2}$$



Superconducting Ham:

$$H = \sum_{\mathbf{p}} \psi_{\mathbf{p}}^\dagger h(\mathbf{p}) \psi_{\mathbf{p}},$$

$$\psi_{\mathbf{p}} = (a_{1,\mathbf{p}}, a_{1,-\mathbf{p}}^\dagger, a_{2,\mathbf{p}}, a_{2,-\mathbf{p}}^\dagger)^T$$

3D Model: symmetries

Impose:

- **Time-reversal** symmetry:

$$C_{\rm TR}^{\dagger} h^*(-\mathbf{p}) C_{\rm TR} = h(\mathbf{p}) \quad C_{\rm TR} = \sigma^y \otimes \mathbb{1}$$

- **Particle-hole** symmetry:

$$C_{\rm PH}^{\dagger} h^*(-\mathbf{p}) C_{\rm PH} = -h(\mathbf{p}) \quad C_{\rm PH} = \mathbb{1} \otimes \sigma^x$$

⇨ Topological superconductor of **type DIII**

3D Model: winding number

$h(\mathbf{p})$: Eigens $|\phi_l(\mathbf{p})\rangle$ and eigenv $E_l(\mathbf{p})$ for $l = 1, ..., 4$

Define:

$$Q(\mathbf{p}) = 2\sum_{l=1,2} |\phi_l(\mathbf{p})\rangle \langle \phi_l(\mathbf{p})| - \mathbb{1} \otimes \mathbb{1} = \begin{pmatrix} 0 & q(\mathbf{p}) \\ q^\dagger(\mathbf{p}) & 0 \end{pmatrix}$$

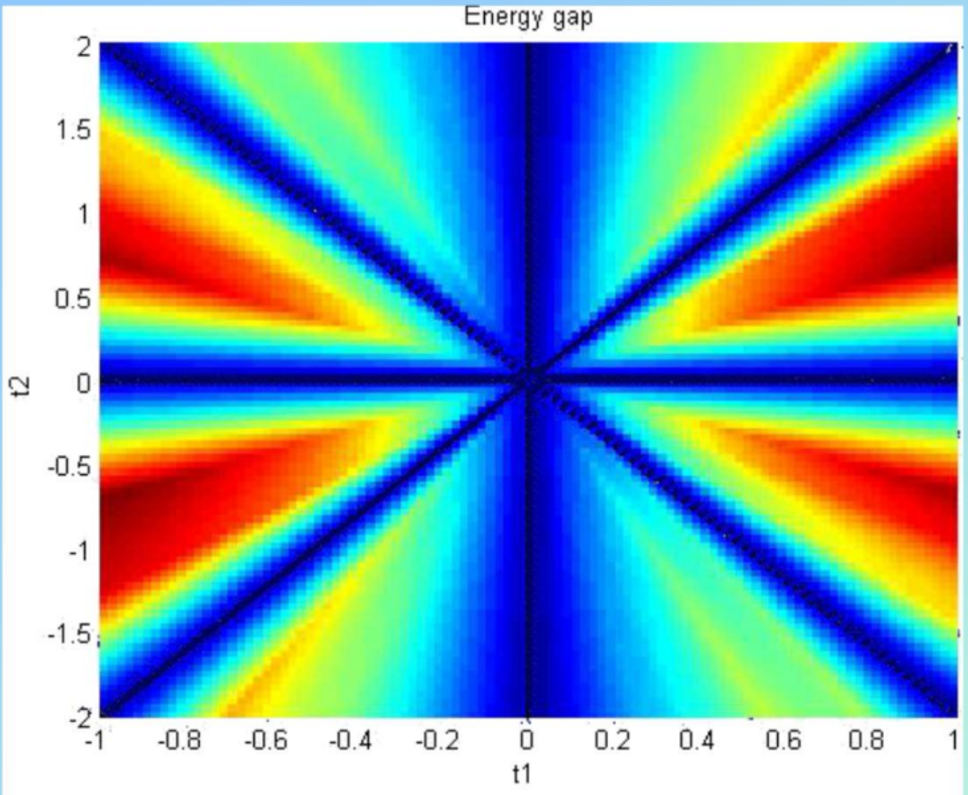
Mapping $T^3 \to S^3$. 3D version of **Chern number**.

$$\nu_{3D} = \frac{1}{24\pi^2} \int_{\mathrm{BZ}} d^3p\, \epsilon^{abc} \mathrm{tr}[(q^{-1}\partial_a q)(q^{-1}\partial_b q)(q^{-1}\partial_c q)]$$

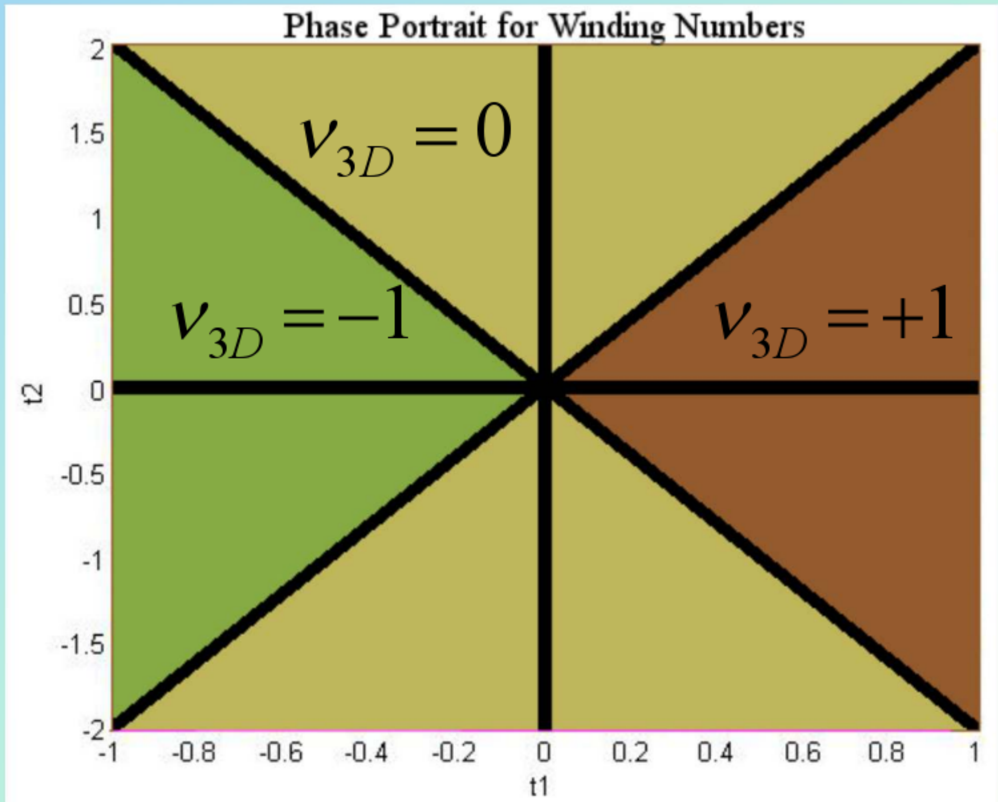
Well defined if $E_2(\mathbf{p}) \neq 0$

3D Model: Topo phase

Energy gap



Winding number

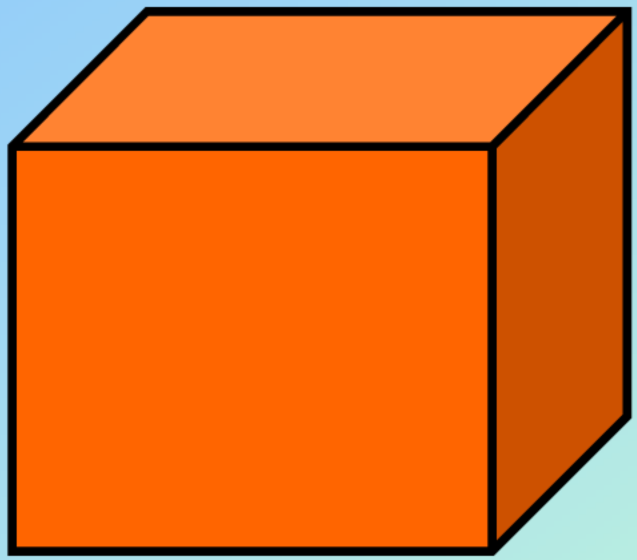


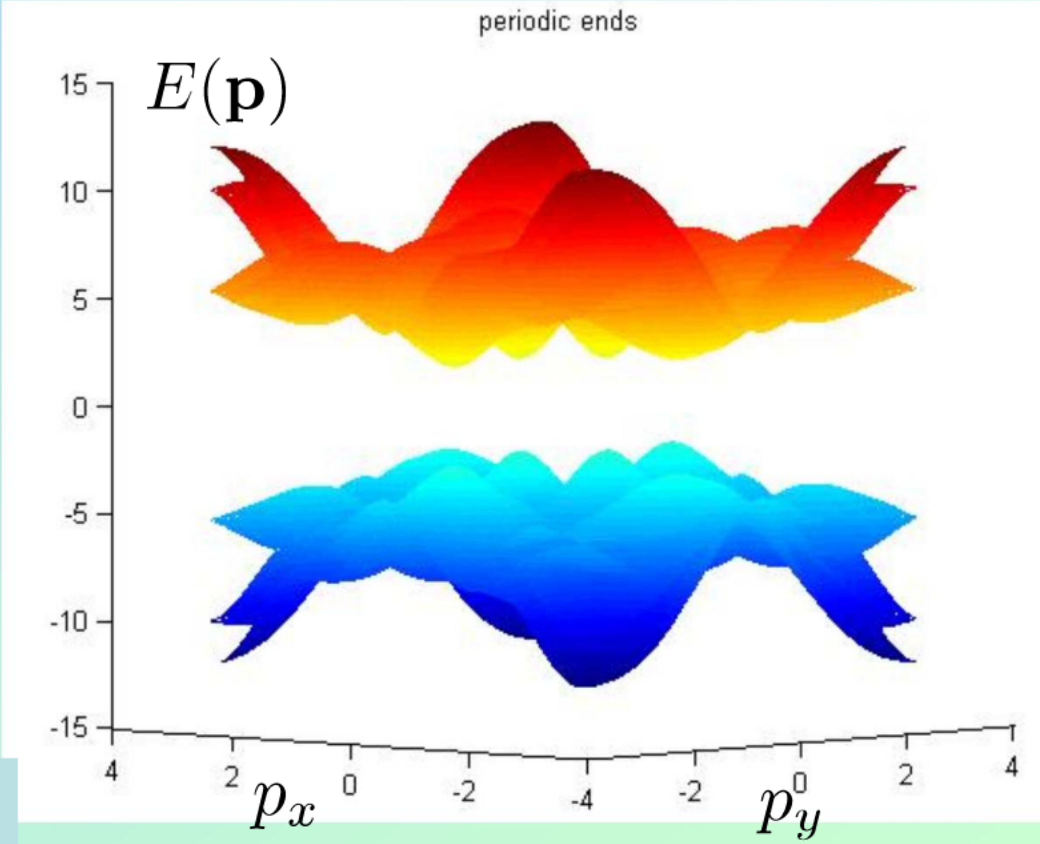
Vary some couplings -> topological phase transitions

3D Model: Topo phase

Periodic BC in all **3 directions**

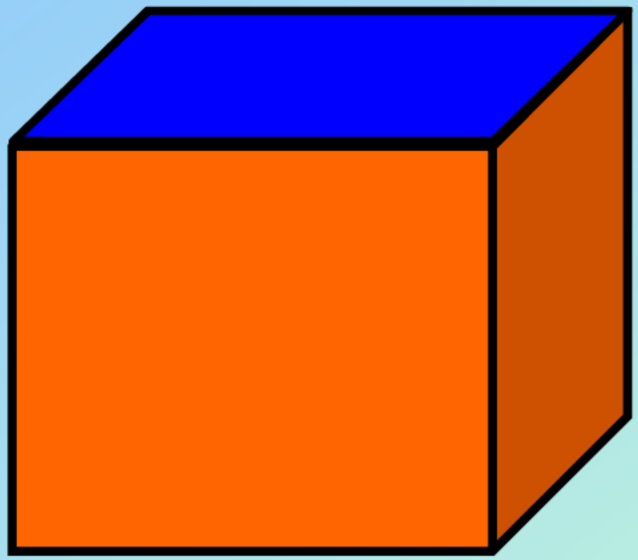
$$\nu_{3D} = 1$$

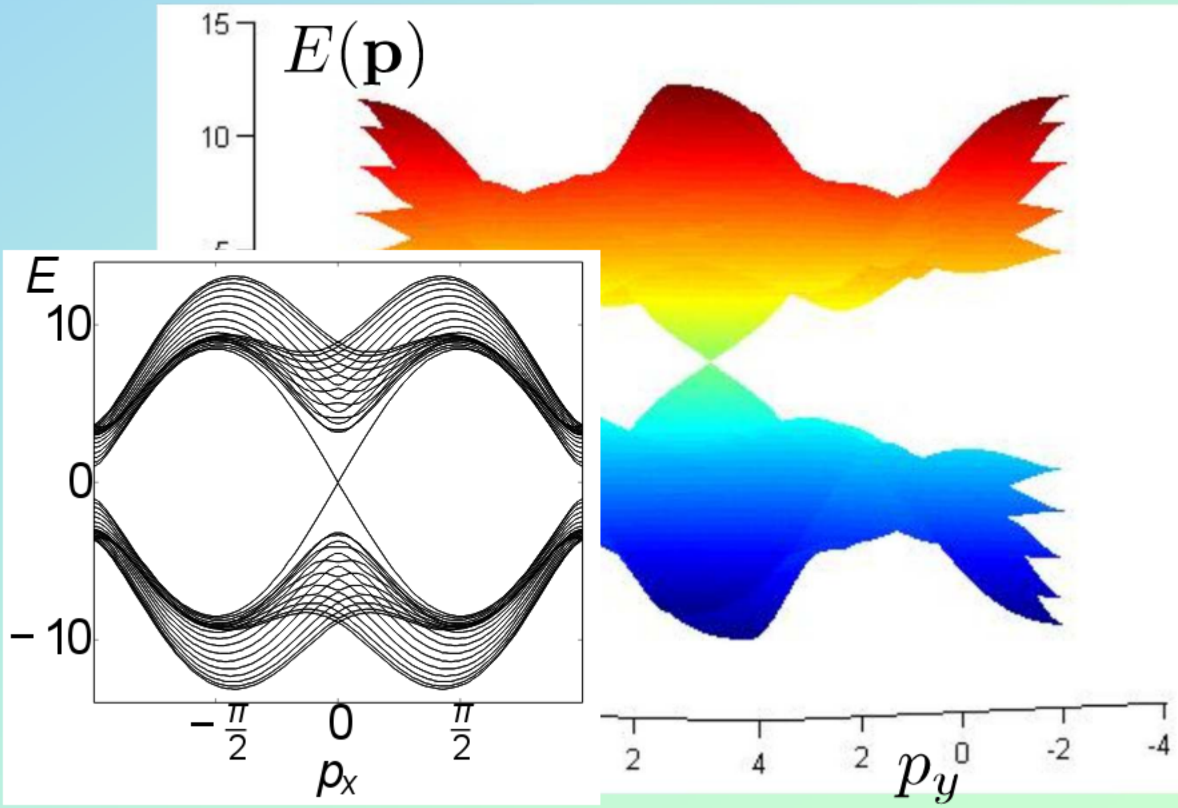




3D Model: Boundaries

Periodic BC in only **2 directions** $\nu_{3D} = 1$





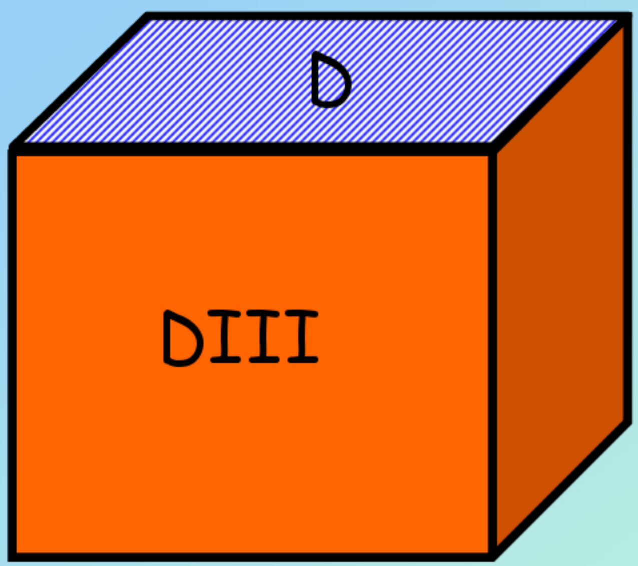
Edge states emerge that manifest themselves as Dirac cones.

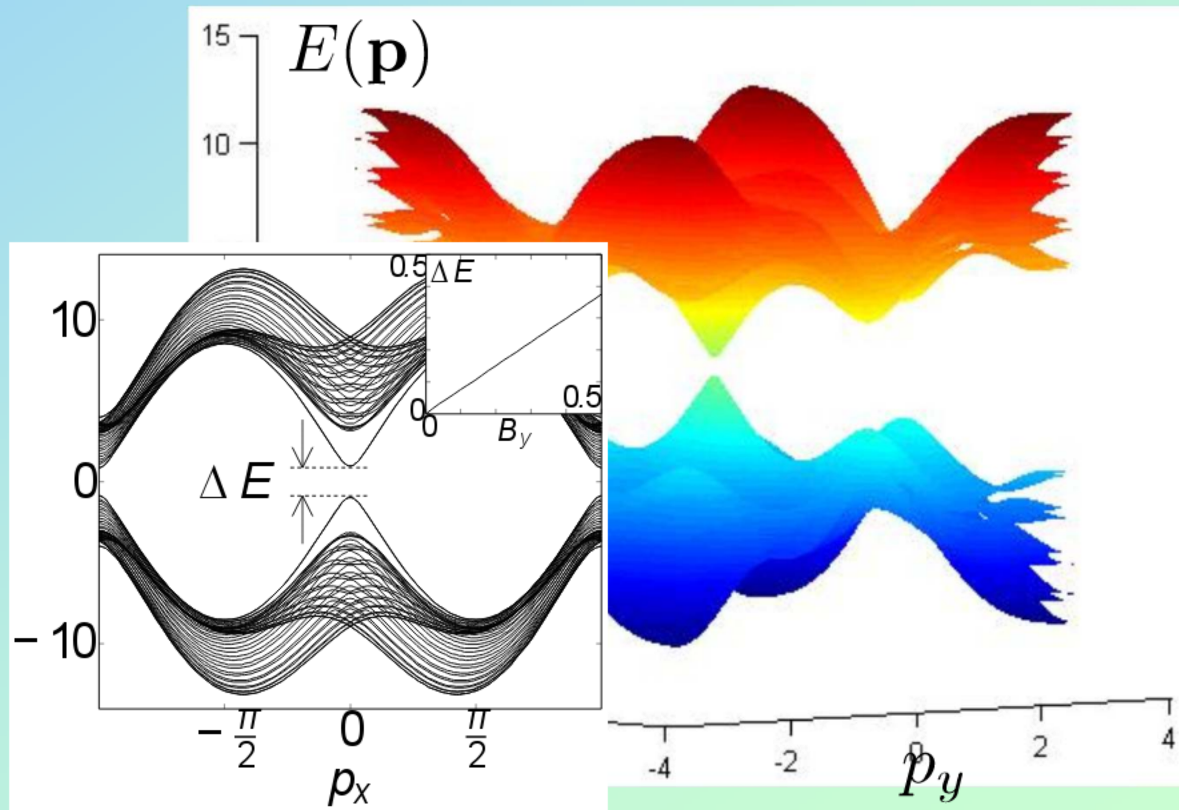
$$\nu = n^L_{\mathrm{Edge}} - n^R_{\mathrm{Edge}}$$

3D Model: Boundaries

Periodic BC in only **2 directions** and boundary interactions that break TR symm.

$$\nu_{3D} = 1$$





Edge states are gapped.

Does the surface support Majorana fermions?

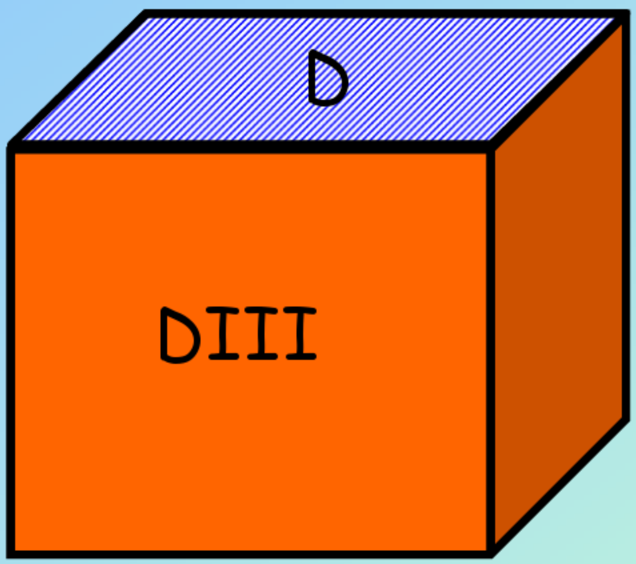
3D Model: Boundaries

Chern number of bulk = Chern number of surface state

$$\nu_{2D} = \nu_{2D}^T + \nu_{2D}^B$$

$$\nu_{3D} = \nu_{2D} = 1$$

$$\nu_{2D}^T = \frac{1}{2}$$



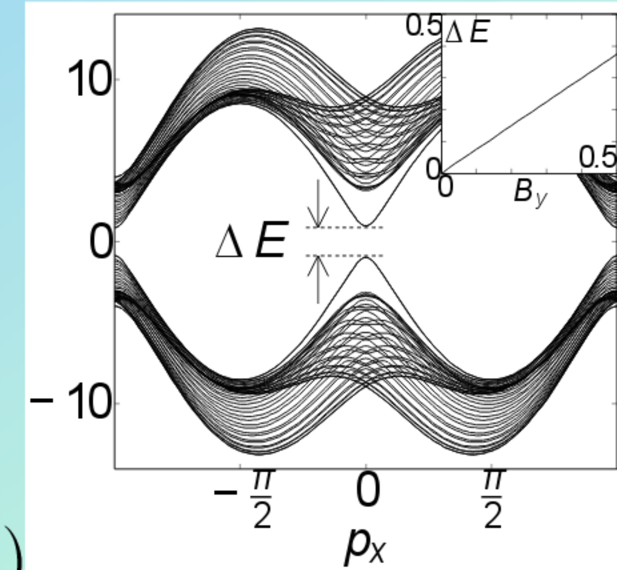
$$\nu_{2D}^B = \frac{1}{2}$$

Topology encoded **non-locally:**

If you shrink bulk to 2D then system is actual type D
-> supports Majoranas

Bulk/boundary locking gives **protection** against **local perturbations**.

3D Model: Bulk-Boundary



Dirac description:

$$S_\psi = \int_M d^4x\, \bar{\psi}(\gamma^\mu \partial_\mu + m)\psi$$

Couple to curvature *integrate fermions*:

$$S_{\mathrm{eff}}^{M,\mathrm{top}} = \frac{1}{2}\frac{\theta}{768\pi^2}\int_M d^4x\, \epsilon^{\mu\nu\alpha\beta}\mathrm{tr}(R^\rho_{\sigma\mu\nu}R^\sigma_{\rho\alpha\beta})$$

R: Riemann tensor $\theta = \nu_{3\mathrm{D}}\pi$

Stokes' theorem:

$$S_{\mathrm{eff}}^{\partial M,\mathrm{top}} = \frac{1}{2}\frac{\theta}{192\pi^2}\int_{\partial M} d^3x\, \epsilon^{\mu\nu\lambda}\mathrm{tr}\left(\omega_\mu\partial_\nu\omega_\lambda + \frac{2}{3}\omega_\mu\omega_\nu\omega_\lambda\right)$$

ω_μ spin connection. TSC of type D with $\nu_{2D} = \frac{\theta}{\pi} = \nu_{3D}$

3D Model: Vortices

Vortex strings have Majoranas at their end points

Majoranas

String tension

Information protected by **string tension:** temperature effects are suppressed, as endpoints attract each other and annihilate (similar to **classical storing of information**)

Conclusions

3D TSCs provide a laboratory for probing new properties of matter.

New physics and **new technological applications**

Lab for generating **stable** Majoranas:

- at surface
- at monopoles in the bulk(?)
- *Stability against finite temperature*