Introduction to Majorana Fermions in solid state systems

K. Sengupta

Department of Theoretical Physics, IACS, Kolkata

Collaborators:
Hyok-Jon Kwon
Victor M. Yakovenko
Moitri Maiti
Rahul Roy

References:
PRB 63 144531 (2001)
EPJB 37 349 (2004)
PRB 74 094505 (2006)
PRL 101 187003 (2008)

Outline

1. ***Hisotry: Fermion number fractionalization in 1+1 D field theory***

2. ***Superconducting platforms for Majorana fermions***

   a) Kitaev chain
   b) Edge states in p-wave superconductors: an exact solution
   c) Other platforms

3. ***Measurement techniques:***
   a) tunneling conductance: Midgap peak
   b) Fractional Josephson effect: even Shapiro step.

4. ***Non-Abelian statistics***

5. ***Conclusion***

Fractionalization in field theory

Fermion number fractionalization in field theory

R.Jackiw and C.Rebbi,
Phys. Rev. D 13, 3398 (1976).

→ *1+1 D coupled field theory of fermions and bosons*

Rajaraman, cond-mat/0103366

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$$

$$\mathcal{L}_B = \frac{1}{2g^2}\left[\left(\frac{\partial\Phi}{\partial t}\right)^2 - \left(\frac{\partial\Phi}{\partial x}\right)^2 - (1/2)(\Phi^2 - 1)^2\right]$$

$$\mathcal{L}_F = \bar{\Psi}\Big(i\partial_\mu\gamma^\mu - m\Phi(x,t)\Big)\Psi$$

The bosonic sector in the absence of the fermions →

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi - \phi + \phi^3 = 0$$

Vacuum sector solution: $\phi(x,t) = \pm 1$

Soliton sector solution: $\phi_S(x) = \pm\, tanh(x/\sqrt{2})$

The soliton sector solution or the kink can not spontaneously decay into the vacuum sector; kinks and anti-kinks can however annihilate each other

Fate of Fermions: Vacuum sector

The Lagrangian in the vacuum sector →

$$\mathcal{L}_F = \bar{\Psi}\Big(i\partial_\mu\gamma^\mu - m\Big)\Psi$$

This leads to the standard Dirac equations in 1+1 D →

$$(-i\alpha\partial_x + \beta m)u_k(x) = E_k\, u_k(x)$$

$$\Big(-i\alpha\partial_x + \beta m)\tilde{u}_k(x) = -E_k\, \tilde{u}_k(x)$$

$$\alpha = \sigma_2 \text{ and } \beta = \sigma_1. \quad E_k = +\sqrt{k^2+m^2}$$

↓

Usual construction of the Fermion Operator in terms of b_k and d_k →

$$\Psi(x,t) = \sum_k [b_k u_k e^{-iE_k t} + d_k^\dagger \tilde{u}_k e^{iE_k t}]$$

$$b_k|vac\rangle = d_k|vac\rangle = 0$$

A consequence of this constriction is that the number (charge) operator always has integer eigenvalues

$$\rho(x,t) = \frac{1}{2}\Big[\Psi^\dagger(x,t)\ ,\Psi(x,t)\Big]$$

$$\begin{aligned} Q &\equiv \int dx\ \rho(x,t) \\ &= \frac{1}{2}\sum_k \Big([b_k^\dagger\ , b_k\ ] + [d_k, d_k^\dagger]\Big) \\ &= \sum_k \Big((b_k^\dagger\ b_k - 1/2) - (d_k^\dagger d_k - 1/2)\Big) \\ &= \sum_k \Big(b_k^\dagger\ b_k - d_k^\dagger d_k\Big) \end{aligned}$$

Q must be an integer.

Note that all factors of ½ cancel due to the existence of paired energy modes

Fate of Fermions: Soliton sector

The Dirac equation now becomes

$$(-\partial_x + m\, tanh \frac{x}{\sqrt{2}})\psi_2 = E\, \psi_1$$
$$(\partial_x + m\, tanh \frac{x}{\sqrt{2}})\psi_1 = E\, \psi_2$$

Apart from the standard paired positive and negative energy solutions with energies E_k and $-E_k$, there is an unpaired localized zero energy mode which is its own charge conjugate

$$\eta_0 = \begin{pmatrix} A\, exp\Big(-m \int^x dy\, tanh\, (y/\sqrt{2})\Big) \\ 0 \end{pmatrix}$$

$$\sigma_3 \eta_0 = \eta_0 .$$

The Fermionic operator now becomes

$$\Psi(x,t) = \sum_{k\neq 0} [b_k \eta_k(x) e^{-iE_k t} + d_k^\dagger \tilde{\eta}_k(x) e^{iE_k t}] + a\eta_0(x)$$

There are two degenerate ground states related to the existence of the zero energy state

$$a|sol\rangle = b_k|sol\rangle = d_k|sol\rangle = 0$$
$$|\hat{sol}\rangle \equiv a^\dagger|sol\rangle \;\; ; \;\; a\,|\hat{sol}\rangle = |sol\rangle$$

These degenerate ground states are distinguished by their charge quantum number

Number fractionalization

$$
\begin{aligned}
Q &\equiv \frac{1}{2}\int dx\Big[\Psi^{\dagger}(x,t)\ ,\Psi(x,t)\Big] \\
&= \frac{1}{2}\sum_k \Big(\ [b_k^{\dagger}\ ,b_k\ ]\ +\ [d_k,d_k^{\dagger}]\Big)\ +\ 1/2[a^{\dagger}\ ,\ a\ ] \\
&= \sum_k \Big(\ (b_k^{\dagger}\ b_k\ -\ 1/2)\ -\ (d_k^{\dagger}d_k - \ 1/2)\Big)\ +\ (a^{\dagger}a\ -\ 1/2) \\
&= \sum_k \Big(b_k^{\dagger}\ b_k\ \ -\ d_k^{\dagger}d_k\Big)\ +\ a^{\dagger}a\ -\ 1/2
\end{aligned}
$$

Number operator now has fractional eigenvalues due to the presence of the bound states

$$Q\ |sol\rangle = -(1/2)|sol\rangle \qquad\qquad Q\ |\hat{sol}\rangle = (1/2)\ |\hat{sol}\rangle$$

Two degenerate ground states have different eigenvalues of number operators.

First example of Fermion number fractionalization arising from degeneracy.

What happens in a real finite solid state sample with N electrons?

Finite size version of the J-R solution

Imagine that the 1+1D field theory is put in a finite size 2L with the periodic boundary condition

$$\psi_{1,2}(-L) \;=\; \psi_{1,2}(L)$$

$$(-\partial_x \;+\; m \tanh \frac{x}{\sqrt{2}} \;)\psi_2 \;=\; E\,\psi_1$$
$$(\partial_x \;+\; m\, tanh \frac{x}{\sqrt{2}} \;)\psi_1 \;=\; E\,\psi_2$$

$$-L \leq x \leq L$$

It turns out that there are now two zero energy states at x=0 and L

$$\eta_0(x) \;=\; \begin{pmatrix} A\; exp\Big(-m \int_0^x dy\; tanh\;(y/\sqrt{2})\Big) \\ 0 \end{pmatrix}$$

$$\tilde{\eta}_0(x) \;=\; \begin{pmatrix} 0 \\ A\; exp\Big(-mL \;+ m \int_0^x dy\; tanh\;(y/\sqrt{2})\Big) \end{pmatrix}$$

Localized at the origin

Localized at one of the edges

The Fermion field now becomes

$$\Psi(x,t) \;=\; \sum_{k\neq 0} [b_k \eta_k(x) e^{-iE_k t} \;+\; d_k^\dagger \tilde{\eta}_k(x) e^{iE_k t}] \;+\; a\eta_0(x) \;+\; c^\dagger\; \tilde{\eta}_0(x)$$

There are now four degenerate ground states which correspond to zero or unit filling of a or c quasiparticles

$$a|sol\rangle = c|sol\rangle = 0 \quad |\widetilde{sol}\rangle = a^\dagger|sol\rangle \,, \quad |\overline{sol}\rangle = c^\dagger|sol\rangle \quad |\widetilde{sol'}\rangle = a^\dagger c^\dagger|sol\rangle$$

There is no fractionalization of the total number: the theory is therefore compatible with integer number of electrons

$$Q = \sum_k \Big((b_k^\dagger b_k - 1/2) - (d_k^\dagger d_k - 1/2) \Big) + (a^\dagger a - 1/2) + (c^\dagger c - 1/2)$$

The effect of fractionalization can still be seen by local probes which will pick up signatures from one of the two states at zero energy.

⇓

Key concept in understanding fractionalization in condensed matter systems

Superconducting Platforms for Majorana Fermions

1D: Kitaev chain

Consider a 1D chain of spinless fermions with the Hamiltonian

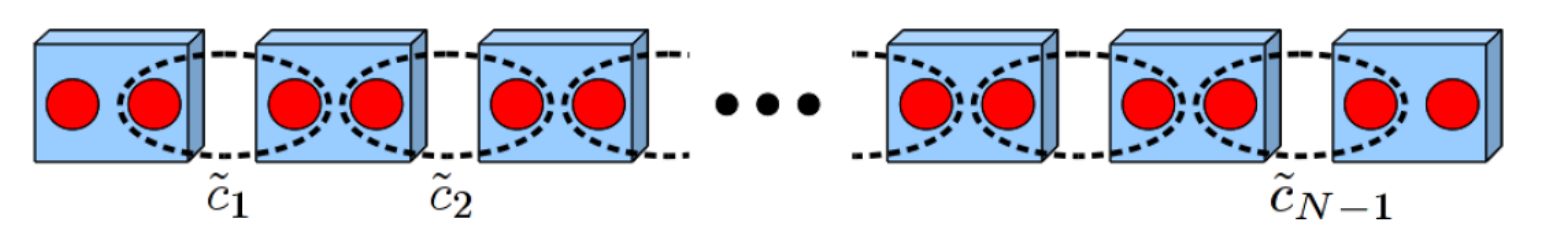
$$\mathcal{H}_{\text{chain}} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left(t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c. \right)$$

Consider this Hamiltonian in the limit $\mu = 0,\, t = \Delta$ ***and define the operators***

$$\gamma_{i,1} = c_i^\dagger + c_i,$$
$$\gamma_{i,2} = i \left(c_i^\dagger - c_i \right)$$

The Hamiltonian can then be expressed as

$$\mathcal{H}_{\text{chain}} = -it \sum_{i=1}^{N-1} \gamma_{i,2} \gamma_{i+1,1}.$$



Ground strate of such a chain shall host two Majorana fermions at its ends.

Superconductivity

Electron-phonon interaction gives rise to effective attractive interaction between electrons of opposite momenta near the Fermi surface.

These electrons can lower their energy by forming bound pairs: Cooper pairs.

The metallic state becomes unstable; new ground-state with well defined phase.

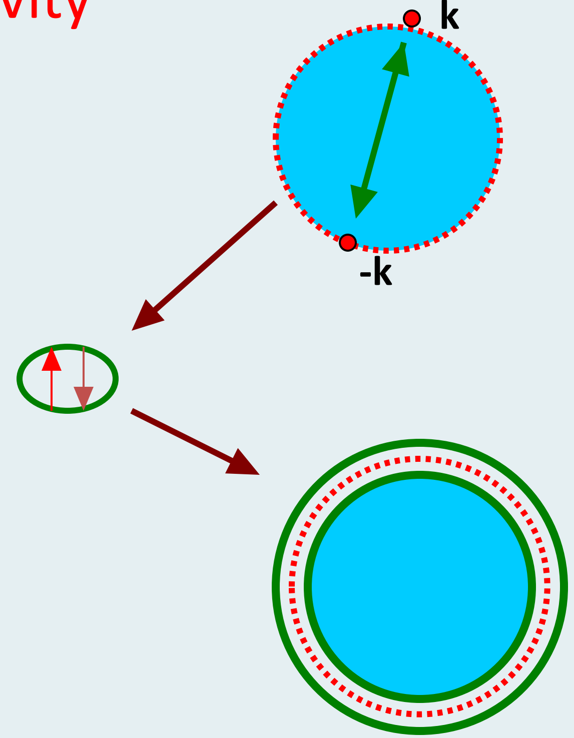
Single particle excitations are Bogoliubov quasiparticles. They are gapped and are linear combination of electrons and holes:

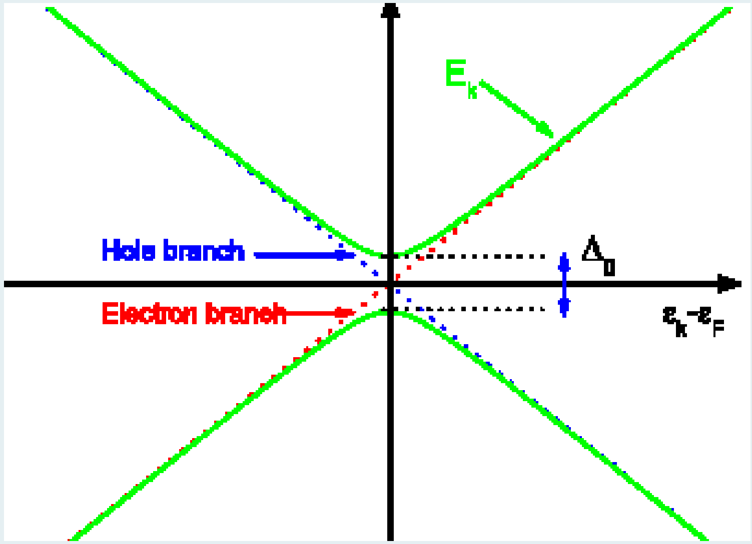
$$\gamma_{k\uparrow} = u_k \psi_{k\uparrow} - v_k \psi^{\dagger}_{-k\downarrow}$$

Quasiparticles obey Bogoliubov-de Gennes (BdG) equations:

$$E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} (\epsilon_k - \epsilon_F) & \Delta \\ \Delta^* & -(\epsilon_k - \epsilon_F) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

Pair Potential





Pair Potential

Mean-field potential due to pairing of electrons

$$\Delta(\mathbf{r_1},\mathbf{r_2}) \sim < \psi(\mathbf{r_1})\psi(\mathbf{r_2}) >$$

Center of Mass coordinate

Relative coordinate

Direction of spin (for triplet pairing)

$$\Delta(\mathbf{k_F}) = \Delta_0 g(\mathbf{k_F})(\sigma \cdot \mathbf{d})\,\mathrm{exp}(i\phi)$$

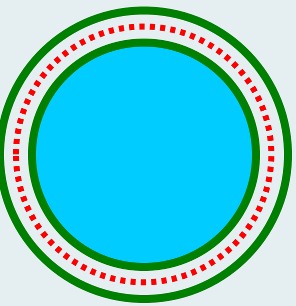
Variation around the Fermi surface

Direction of spin (for triplets only)

Global phase factor

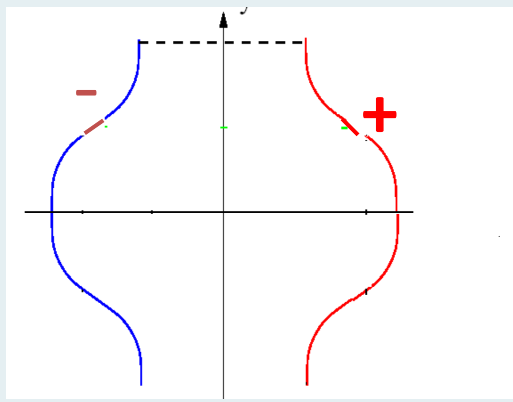
Variation of pair-potentials around the Fermi surface

$g(\mathbf{k}) = 1$



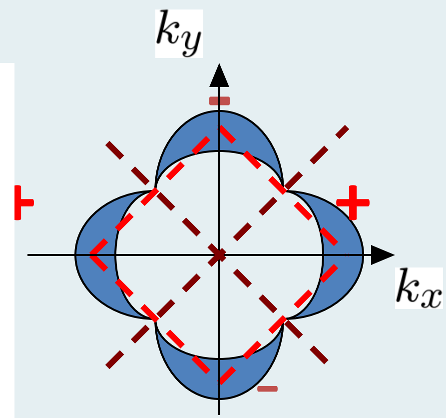
Standard s-wave

$g(\mathbf{k}) = k_x/k_F$



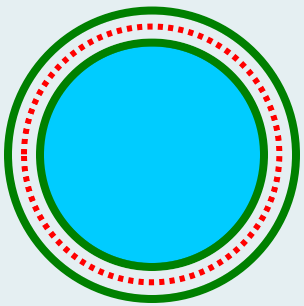
Triplet p-wave in Q1D organic materials

$g(\mathbf{k}) = (k_x^2 - k_y^2)/k_F^2$



d-wave in cuprates

$g(\mathbf{k}) = (k_x + ik_y)/k_F$

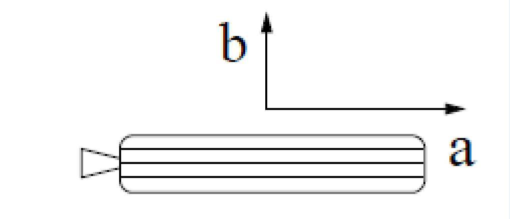


Triplet chiral p-wave in Ruthenates

However, note additional momentum-dependent phase

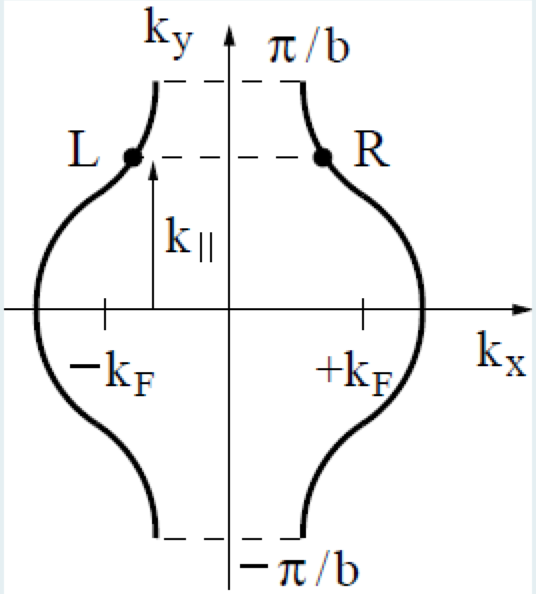
Edge states in TMTSF

(TMTSF)$_2$ X is an organic anisotropic metal with dispersion

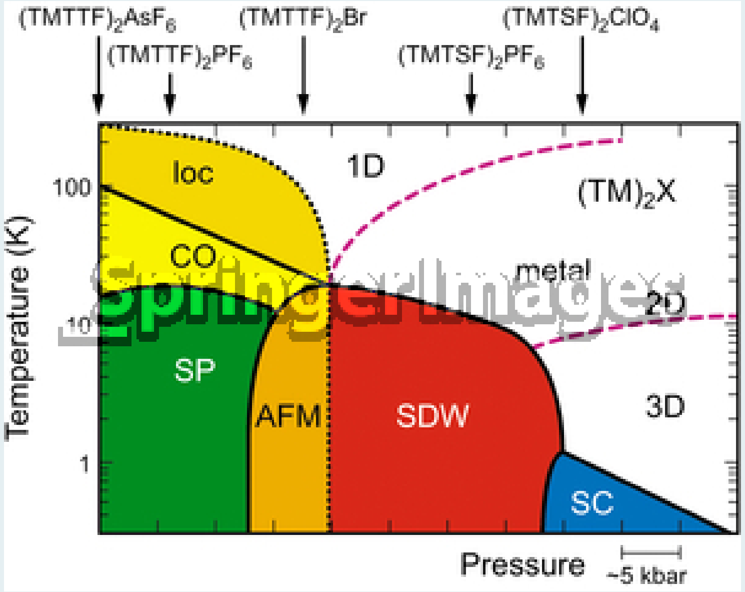


$$\epsilon(\mathbf{k}) = v_F(|k_x| - k_F) - 2t_b\cos(k_y b) - 2t_c\cos(k_z c).$$

where $v_F = 2t_a\, a/h$ and $t_a >> t_b >> t_c$ leading to quasi-1D nature of the compound



Fermi surface of TMTSF



Under optimal pressure, the compound undergoes superconducting transition around 1-2K . Experiments seem to suggest triplet superconductivity (no change in Knight shift; H_{c2} exceeds Pauli limit by a factor of 4 etc)

Triplet Superonductivity in TMTSF

The pair potential for triplet superconductivity is given by

$$\langle \hat{\psi}^{\alpha}_{\sigma} \hat{\psi}^{\bar{\alpha}}_{\sigma'} \rangle \propto i\hat{\sigma}^{(y)}(\mathbf{d} \cdot \hat{\boldsymbol{\sigma}})\Delta^{\alpha}$$

$$\Delta^{\mathrm{R}} = -\Delta^{\mathrm{L}}$$

α=R,L.

The BdG equation for the quasiparticles is given by

$$\begin{pmatrix} -i\alpha v_{\mathrm{F}}\partial_x & (\hat{\boldsymbol{\sigma}} \cdot \mathbf{d})\, \Delta^{\alpha}(x) \\ (\hat{\boldsymbol{\sigma}} \cdot \mathbf{d})\, \Delta^{\alpha *}(x) & i\alpha v_{\mathrm{F}}\partial_x \end{pmatrix} \begin{pmatrix} u^{\alpha}_{n} \\ v^{\alpha}_{n} \end{pmatrix} = E_n \begin{pmatrix} u^{\alpha}_{n} \\ v^{\alpha}_{n} \end{pmatrix},$$

where $\alpha v_{\mathrm{F}} = \pm v_{\mathrm{F}}$ for α=R,L.

Experimental inputs suggests that d is a real vector pointing along a; we choose our spin quantization along d leading to opposite spin-pairing.

These are described by a 2 component matrix equation

$$\begin{pmatrix} -i\alpha v_{\mathrm{F}}\partial_x & \sigma\Delta^{\alpha}(x) \\ \sigma\Delta^{\alpha *}(x) & i\alpha v_{\mathrm{F}}\partial_x \end{pmatrix} \begin{pmatrix} u^{\alpha}_{n,\sigma} \\ \sigma v^{\alpha,\bar{\sigma}}_{n} \end{pmatrix} = E_n \begin{pmatrix} u^{\alpha}_{n,\sigma} \\ \sigma v^{\alpha,\bar{\sigma}}_{n} \end{pmatrix}.$$

- *is determined by the self-consistency condition in terms u_n and v_n*

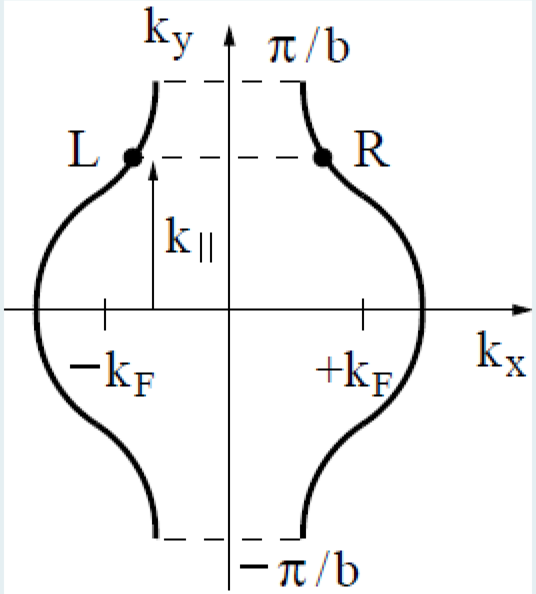
$$\Delta^{\alpha}(x) = g \sum_n u^{\alpha}_{n}(x) v^{\alpha *}_{n}(x)$$

The edge problem

Consider a semi-infinite sample occupying x>0 having an impenetrable edge at x=0

Upon reflection from such an edge, the BdG quasiparticles gets reflected from L To R on the Fermi surface

The right and the left moving quasiparticles see opposite sign of the pair-potential



The BdG wavefunction is superposition of the left and the right moving quasiparticles

$$\Psi = \frac{1}{\sqrt{2}}\left[e^{i\mathbf{r}\cdot\mathbf{k}_F^R}\begin{pmatrix}u_n^R(x)\\ v_n^R(x)\end{pmatrix} - e^{i\mathbf{r}\cdot\mathbf{k}_F^L}\begin{pmatrix}u_n^L(x)\\ v_n^L(x)\end{pmatrix}\right]$$

The boundary condition for the impenetrable edge

$$\Psi(x=0)=0 \quad\Longrightarrow\quad u^R(0)=u^L(0), \qquad v^R(0)=v^L(0).$$

Exact solution [KS, I. Zutic, H-J Kwon, V. Yakovenko and S. Das Sarma, PRB 2001]

1. Extend the wavefunction from positive semispace to the full space using the mapping

$$[u(x), v(x)] = [u^{\mathrm{R}}(x), v^{\mathrm{R}}(x)] \text{ and } \Delta(x) = \Delta^{\mathrm{R}}(x) \qquad x > 0$$

$$[u(x), v(x)] = [u^{\mathrm{L}}(-x), v^{\mathrm{L}}(-x)] \text{ and } \Delta(x) = \Delta^{\mathrm{L}}(-x) \qquad x < 0$$

2. This leads to a single BdG equation defined for all x

$$\begin{pmatrix} -iv_{\mathrm{F}}\partial_x & \Delta(x) \\ \Delta^*(x) & +iv_{\mathrm{F}}\partial_x \end{pmatrix} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = E_n \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix},$$

$$\Delta(x) = g \sum_n u_n(x) v_n^*(x). \qquad -\infty < x < \infty$$

3. The boundary condition of the edge problem translates to continuity of u and v at x = 0

4. For p-wave, D(x) changes sign at the origin and the problem is exactly mapped onto the 1D CDW problem solved by SSH and Brazovski (JETP 1980)

5. This allows us to write down the exact self-consistent solution for the edge problem

$$\Delta(x) = i\Delta_0 \tanh(\kappa x);$$

$$E_0 = 0, \quad \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} = \frac{\sqrt{\kappa}}{2\cosh(\kappa x)} \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$E_k = \pm\sqrt{v_{\mathrm{F}}^2 k^2 + \Delta_0^2},$$

$$\begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix} = \frac{e^{ikx}}{2E_k\sqrt{L_x}} \begin{pmatrix} E_k + v_{\mathrm{F}}k + \Delta(x) \\ E_k - v_{\mathrm{F}}k - \Delta(x) \end{pmatrix},$$

Properties and spin response of the edge states

The edge states carry zero net charge

$$\rho \propto |u_0|^2 - |v_0|^2 = 0.$$

The edge states with momenta k_y and $-k_y$ are identical ⟹

$$\hat{\Psi}^{\dagger}_{\mathbf{k}_{\parallel},\sigma} = \pm \hat{\Psi}_{-\mathbf{k}_{\parallel},\bar{\sigma}}$$

They have half the number of modes and thus have fractional eigenvalues ⟹ There is one Fermion state for each (k_y,$-k_y$) pair per spin

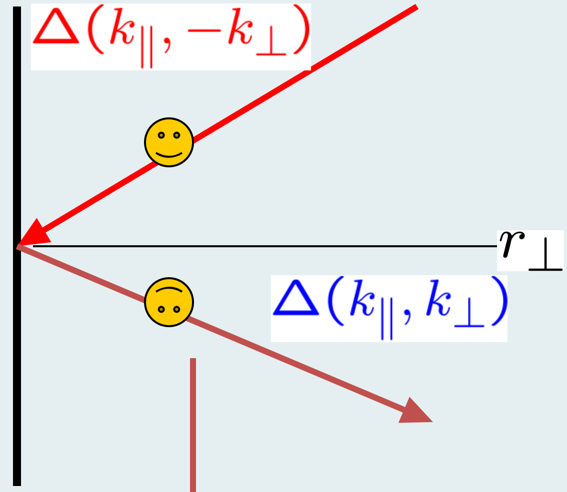
⇓

In the presence of a Zeeman field, one generate a magnetic field of $m_B/2$ per chain end. This is formally equivalent to having $S_z=h/4$ for these states. ⟶

$$E_0 = \mp \mu_B H.$$

These states would be Majorana Fermions in 1D and for spinless (or spin-polarized) Fermions with equal-spin pairing (current research focus)

Presence of the edge



- Solve the BdG equations with the edge boundary condition:

$$\psi(x = 0) = 0$$

Cuprates
(Hu 1992, Adigali *et al*., 1998)

Q1D organic superconductors
(Sengupta *et al*., 2001, Kashiwaya *et al*., 2002)

Ruthenates
(Sigrist *et al*., 2001, Sengupta *et al*., 2002)

Quasiparticles see a momentum-dependent change of phase $\phi(k_{\parallel})$ of the pair-potential upon reflection from the edge

Additional localized states within the gap with energy E

$$E = |\Delta_0 g(k_{\parallel})| \cos\left[\phi(k_{\parallel})/2\right]$$

$$\kappa = \left(|\Delta_0 g(k_{\parallel})|^2 - E(k_{\parallel})^2\right)^{1/2} / \hbar v_F$$

How do we find the phase $\phi(k_{\parallel})$?

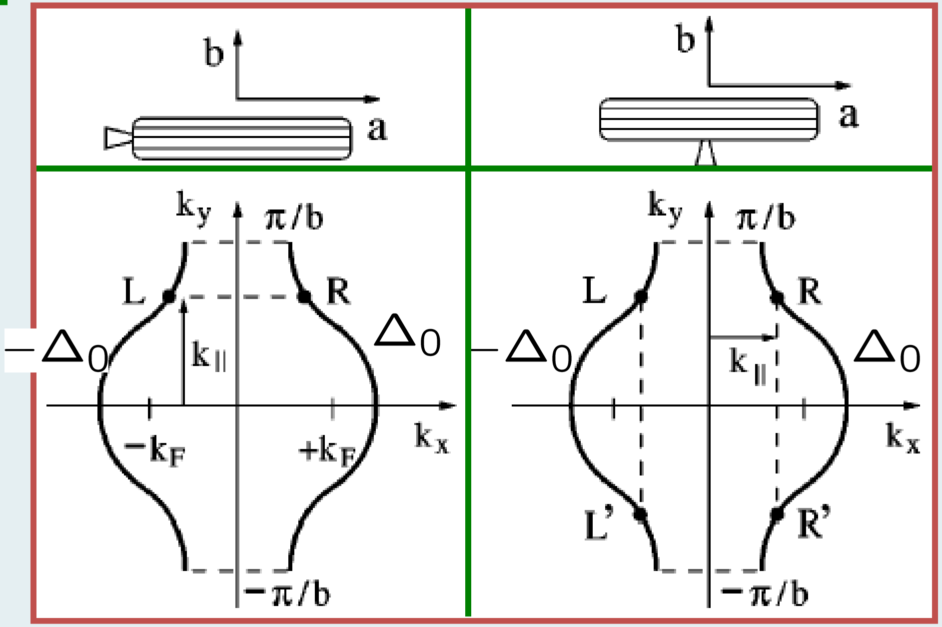
Q1D organic superconductors

Quasiparticles, upon reflection from the edge, sees opposite sign of the pair potential

$\phi(k_y) = \pi$

$E \sim \cos[\phi(k_y)/2] = 0$

Localized midgap states at this edge



Quasiparticles, upon reflection from the edge, sees same sign of the pair potential

$\phi(k_x) = 0, 2\pi$

$E = \pm\Delta_0$

No subgap states at this edge

Creating artificial platforms for Majorana Fermions

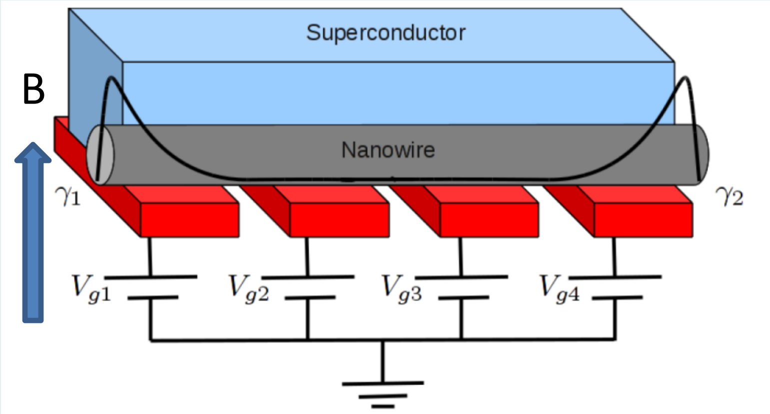
Proximity induced effective p-wave: 1D nanowire

Schematic setup for proximity induced superconductors

$$\mathcal{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \int d^D r \; \Psi^\dagger_\sigma(\mathbf{r}) H_0(\mathbf{r}) \Psi_\sigma(\mathbf{r}),$$

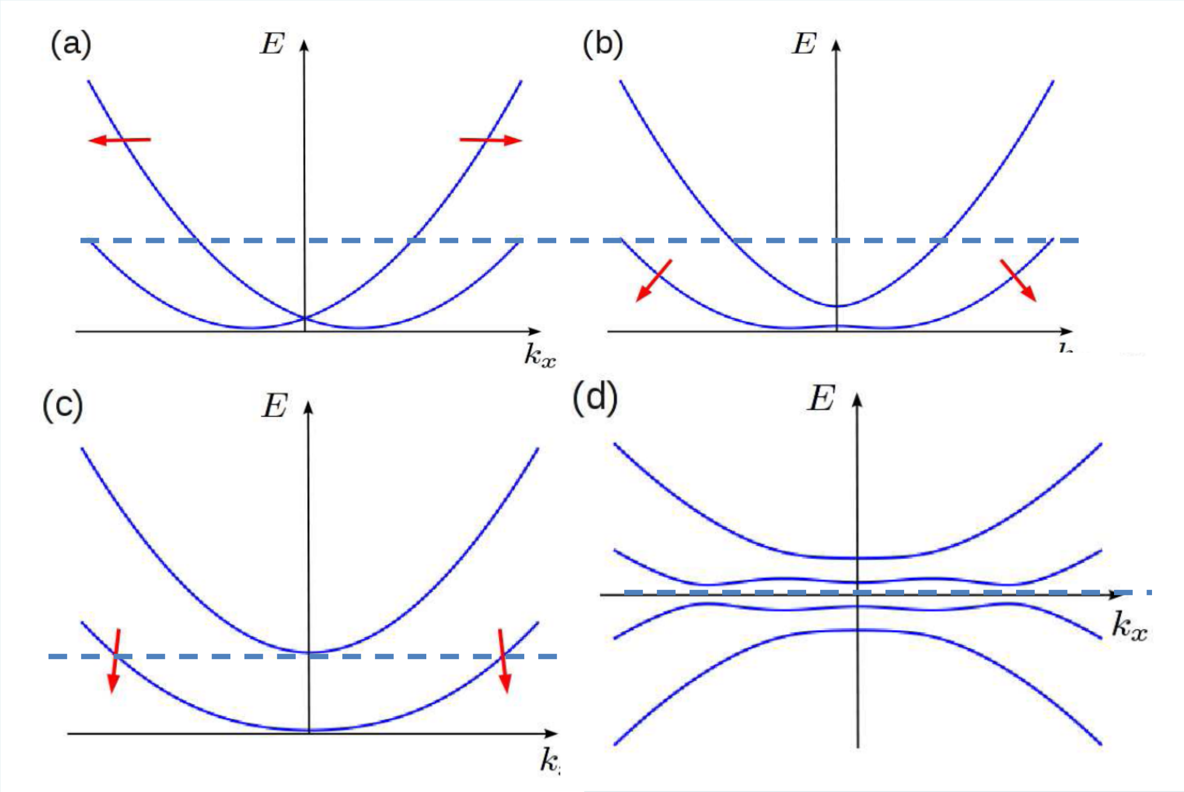
$$H_0(x) = \frac{k_x^2}{2m} - \mu + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z$$

$$\mathcal{H}_S = \int d^D r \; d^D r' \; \Psi_\downarrow(\mathbf{r}) \Delta(\mathbf{r}, \mathbf{r}') \Psi_\uparrow(\mathbf{r}')$$



Band structure and formation of p-wave superconductor

Realization of p-wave superconductor in the band basis and hence Majorana Fermions at the edge

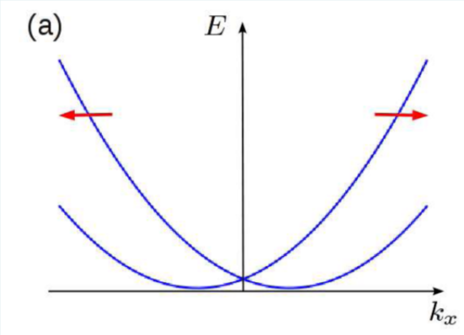


A bit more details

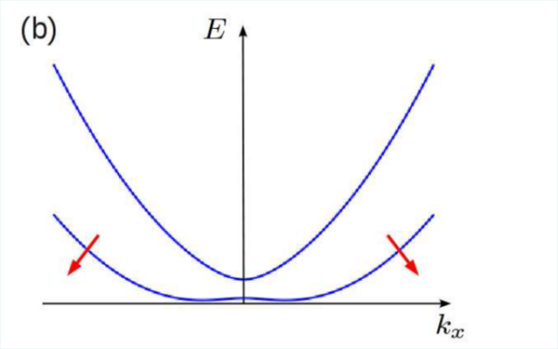
The Hamiltonian of the wire in the absence of the superconductor can be easily diagonalized

$$H_0(x) = \frac{k_x^2}{2m} - \mu + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z \quad \longrightarrow \quad E_\pm(k_x) = \frac{k_x^2}{2m} - \mu \pm \sqrt{(\tilde{\alpha}k_x)^2 + \tilde{B}^2},$$

Without the magnetic field, the spin-orbit coupling shifts the bands in opposite direction



With small B, the zero energy crossing turns into an anticrossing



Spin-orbit also makes the spin direction momentum dependent; thus with larger B when only the lower band is occupied and the fermi energy is in the gap, a proximate s-wave superconductor can induce effective p-wave superconductivity

Such a superconductivity occurs if

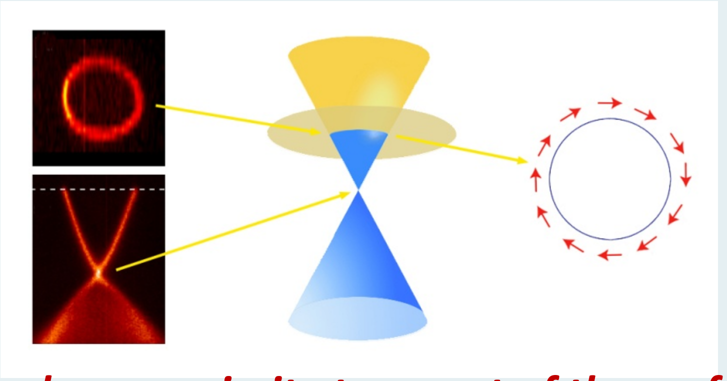
$$|\tilde{B}| > \sqrt{\Delta^2 + \mu^2}. \quad \longrightarrow$$ Gapped spectrum for all k

$$|\tilde{B}| > |\mu|. \quad \longrightarrow$$ Fermi energy in the gap

Proximity induced superconductivity on a surface of a strong TI

Surface of a strong TI hosts a single Dirac cone

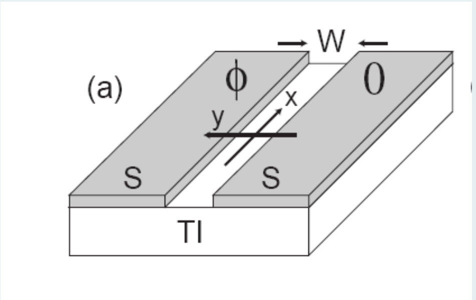
One has one state per momenta with a definite spin direction fixed by helicity.



Idea of Fu-Kane: bring in a s-wave superconductor in close proximity to a part of the surface

$$\mathcal{H} = -iv\tau^z \sigma \cdot \nabla - \mu\tau^z + \Delta_0(\tau^x \cos\phi + \tau^y \sin\phi). \Longrightarrow E_{\mathbf{k}} = \pm\sqrt{(\pm v|\mathbf{k}| - \mu)^2 + \Delta_0^2}.$$

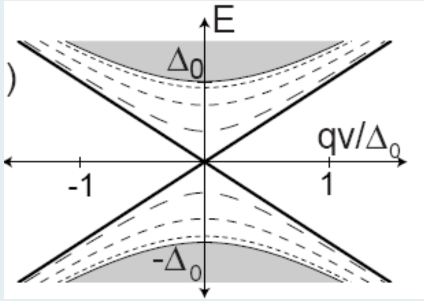
For $\mu \gg \Delta_0$, the low energy spectrum represents a p+ip superconductor



To see this, note that by choosing

$$c_{\mathbf{k}} = (\psi_{\uparrow\mathbf{k}} + e^{i\theta_{\mathbf{k}}}\psi_{\downarrow\mathbf{k}})/\sqrt{2}$$

$$\mathbf{k} = k_0(\cos\theta_{\mathbf{k}}, \sin\theta_{\mathbf{k}}) \text{ and } vk_0 \sim \mu$$



One gets an effective p+ip Hamiltonian

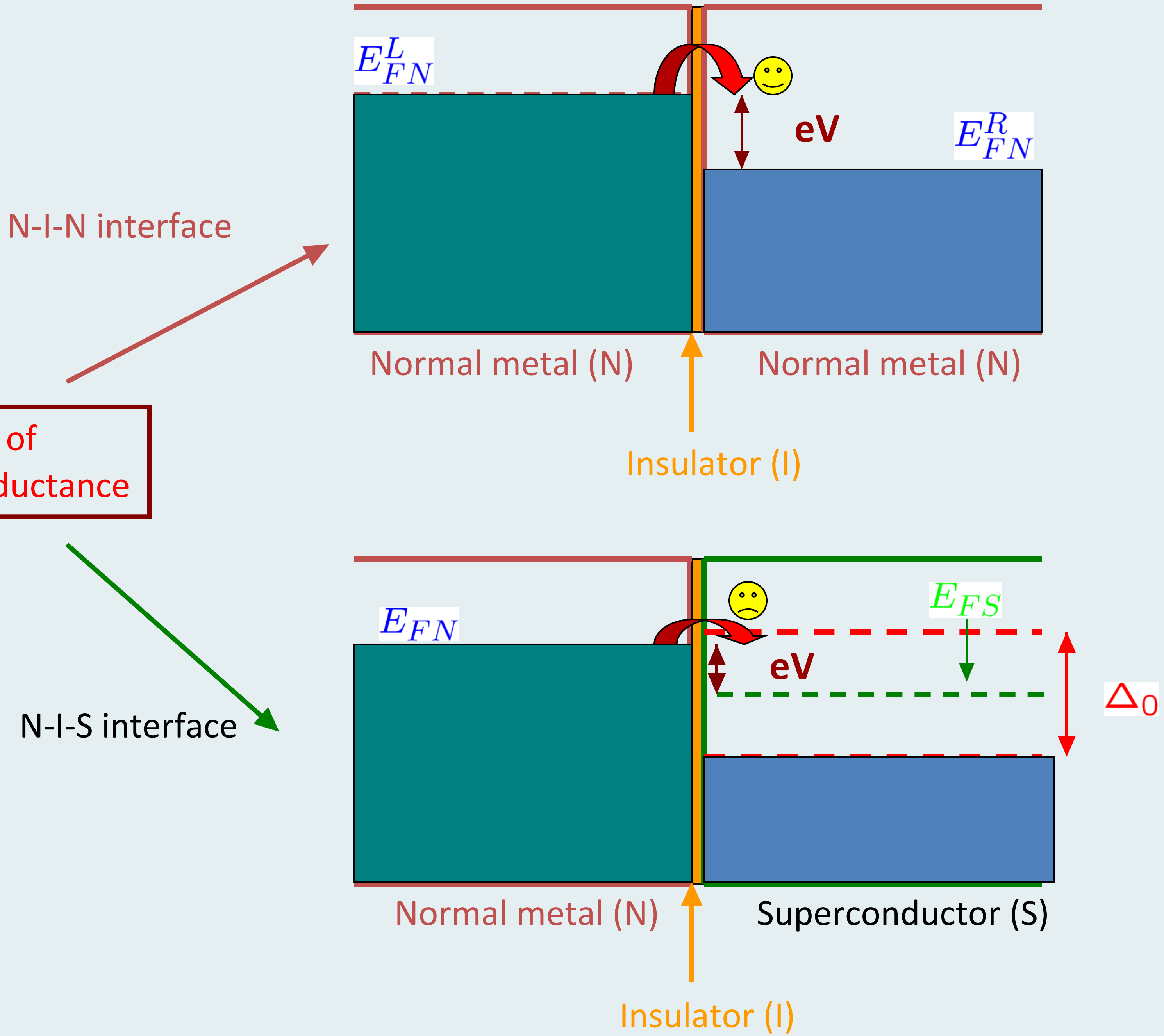
$$\sum_{\mathbf{k}}(v|\mathbf{k}| - \mu)c^\dagger_{\mathbf{k}}c_{\mathbf{k}} + (\Delta e^{i\theta_{\mathbf{k}}}c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}} + h.c.)/2.$$

Majorana modes appear at the chiral interface or vortex centers of such superconductors (Ivanov '03)

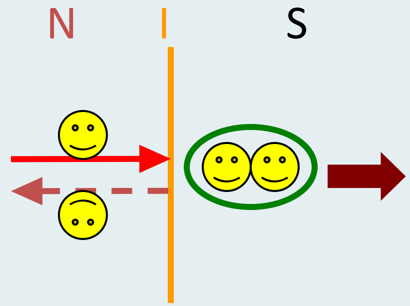
Detection of Majorana states: tunneling conductance

Experiments: How to look for edge states

Measurement of tunneling conductance



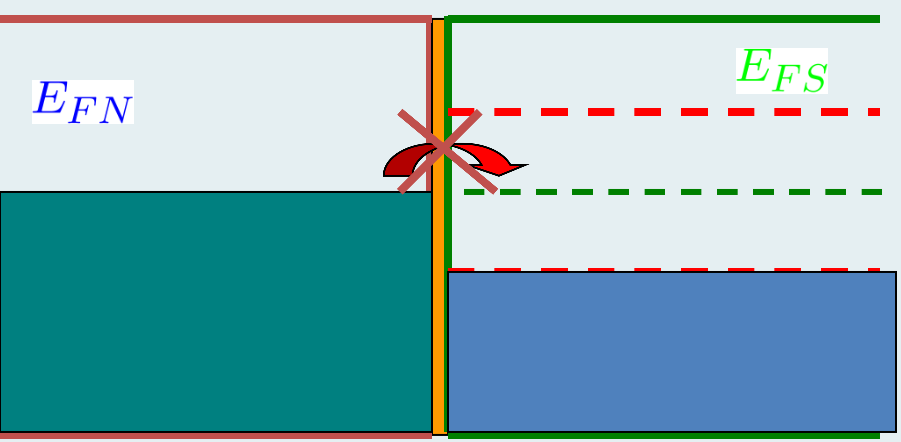
Basic mechanism of current flow in a N-I-S junction



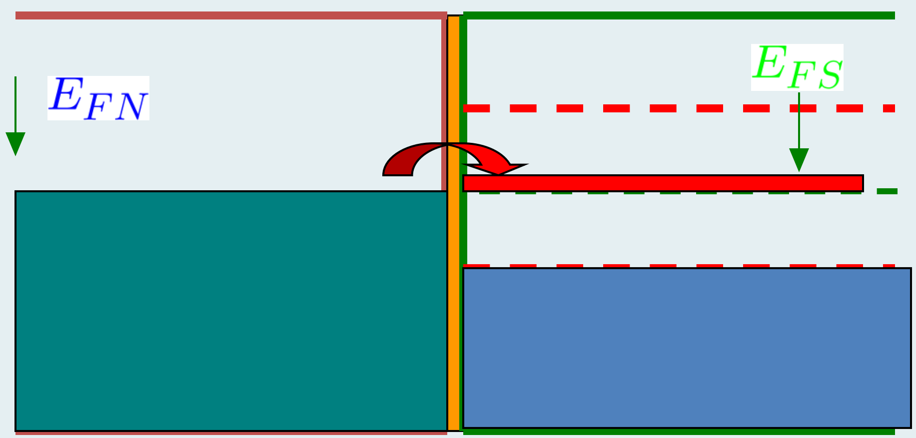
Andreev reflection
2e charge transfer

Strongly suppressed if the insulating layer provides a large potential barrier: so called tunneling limit

In the tunneling limit, the tunneling conductance carries information about the density of quasiparticle states in a superconductor.



Edges with no Midgap States

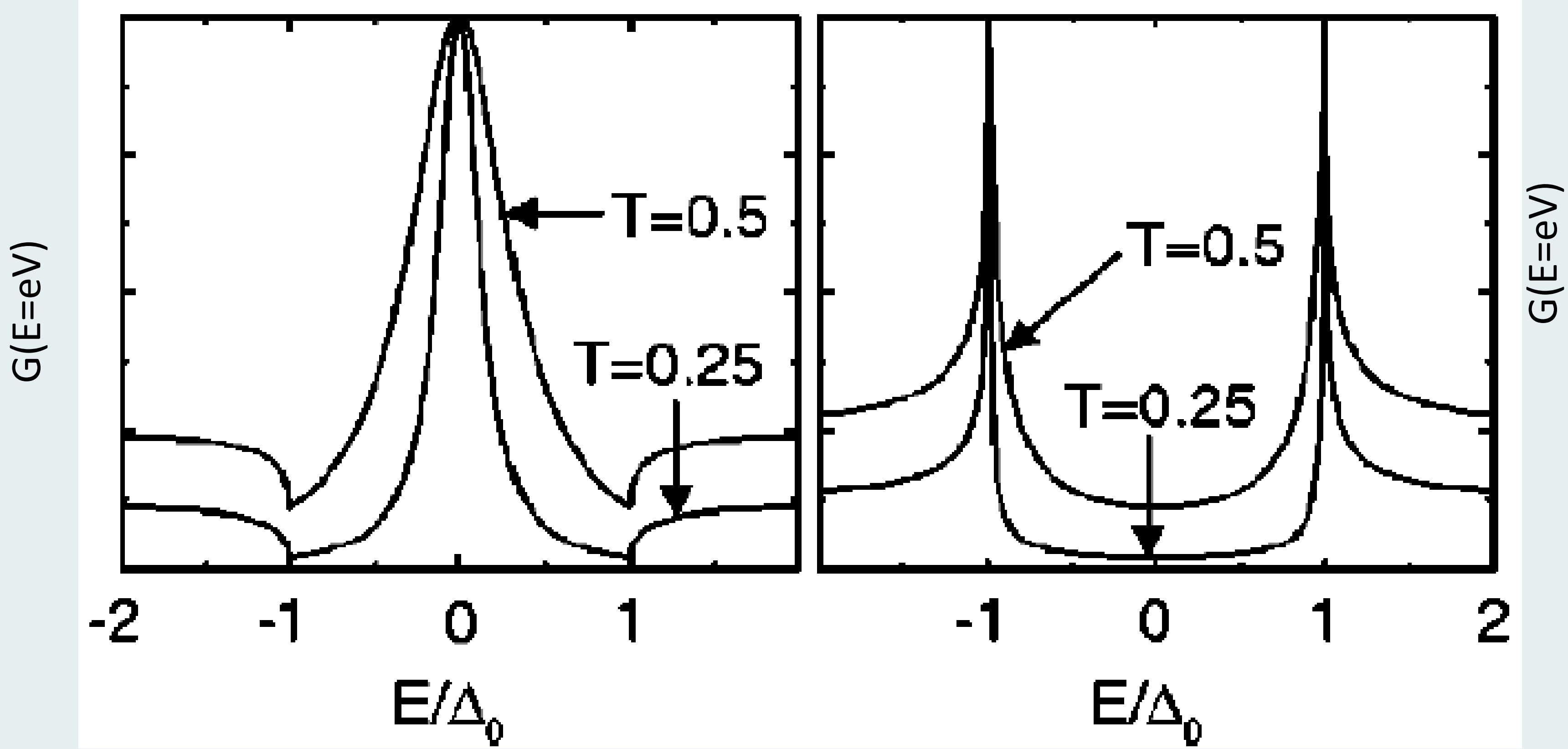


Edges with Midgap States

Typical tunneling conductance Curves

$$G_-(E) = \frac{T^2/2(1-T)}{(E/\Delta_0)^2 + T^2/4(1-T)}, \quad |E| \leq \Delta_0.$$

$$G_+(E) = \frac{T^2/2(1-T)}{1-(E/\Delta_0)^2 + T^2/4(1-T)}, \quad |E| \leq \Delta_0.$$



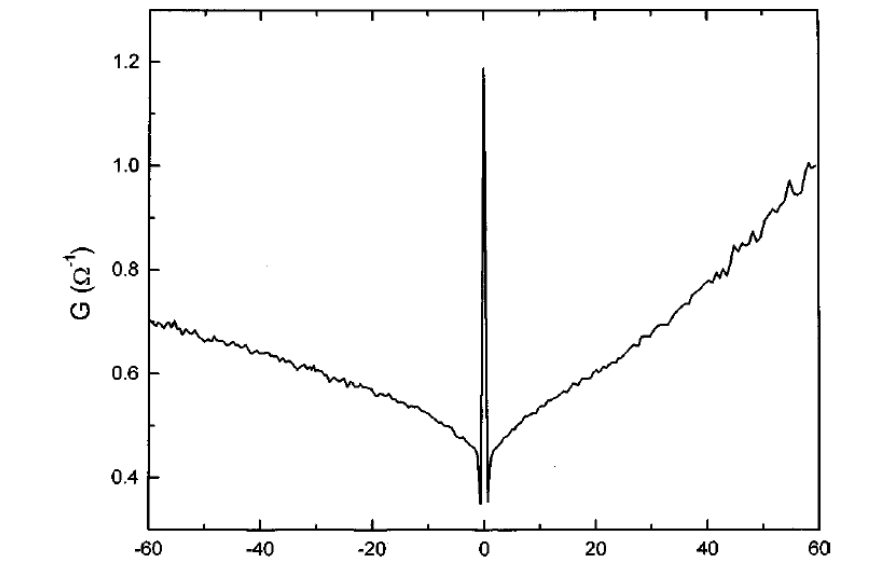
Edge with midgap states.

Edge without midgap states.

Experiments for cuprates and TMTSF

Cuprates

Covington *et.al*. 1997, Krupke and Deuscher 1999 ………

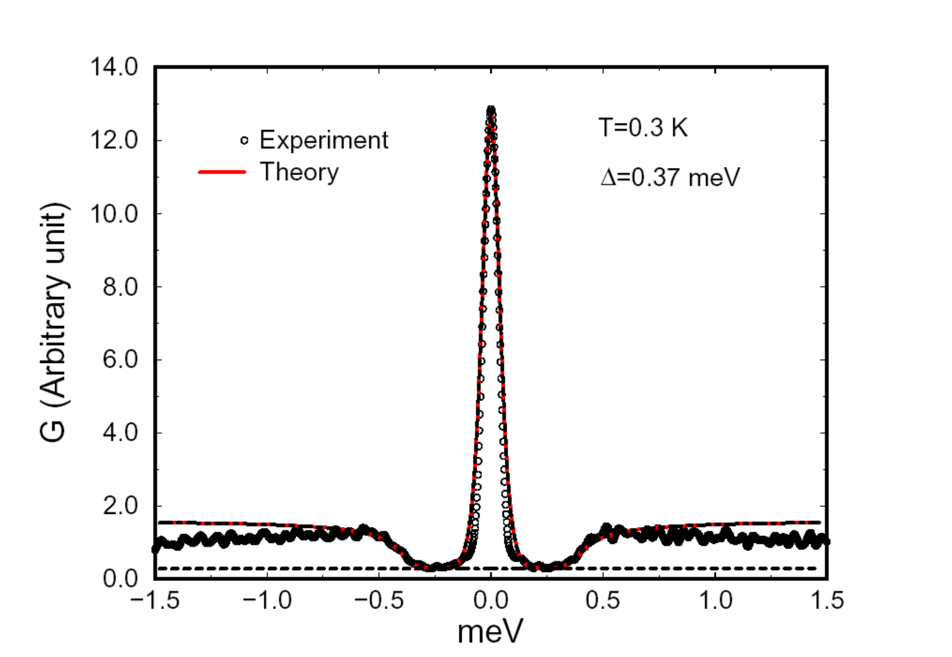


mev

Data from Cucolo *et al*, 2000.
Tunneling in a-b plane in YBCO

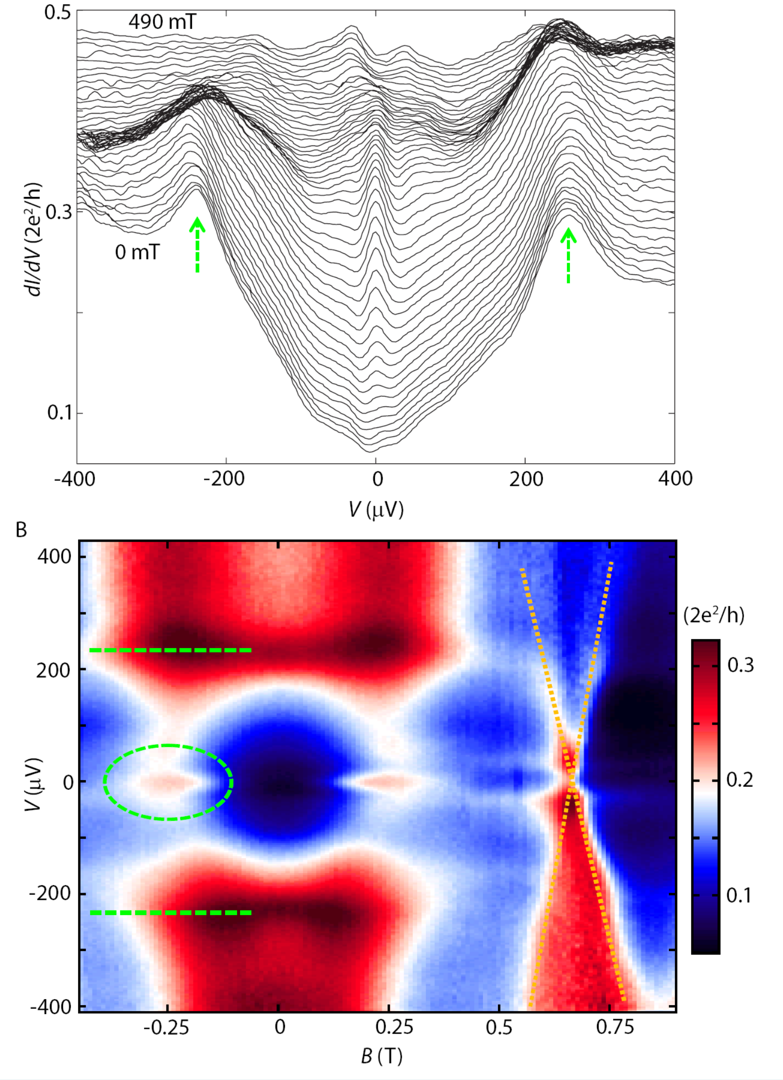
TMTSF

Naughton *et al*., unpublished



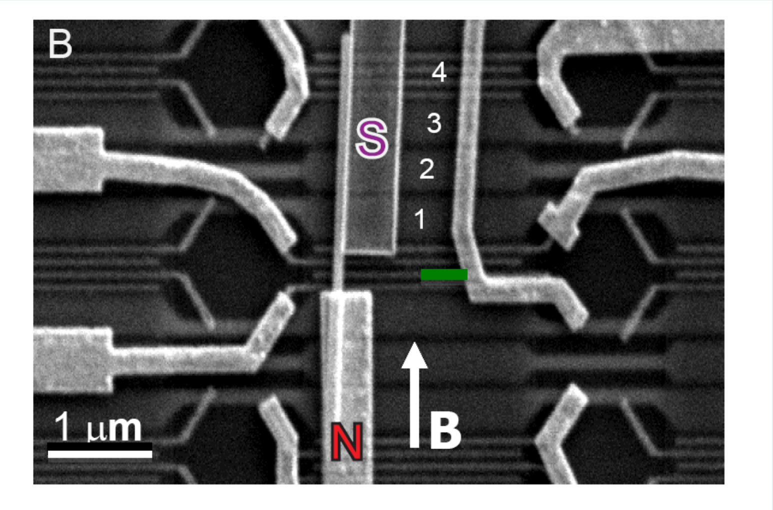
Unpublished data from Naughton *et.al*

Midgap state in 1D nanowire junction

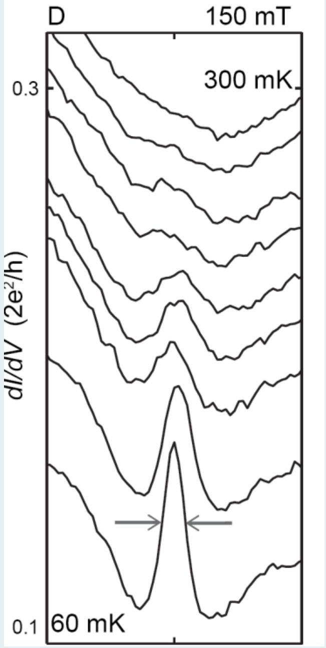


Field and voltage dependence of the zero-bias peak

Experimental setup schematics



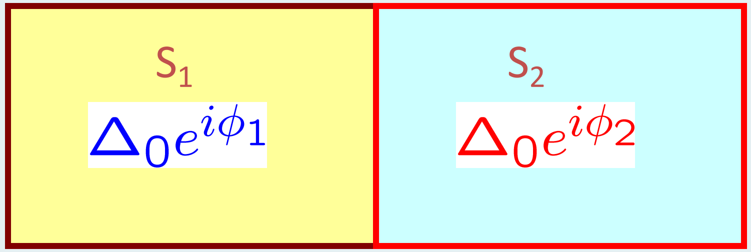
Temperature dependence of the midgap peak



Why is the midgap peak so small??

Signature of Majorana in Josephson effect

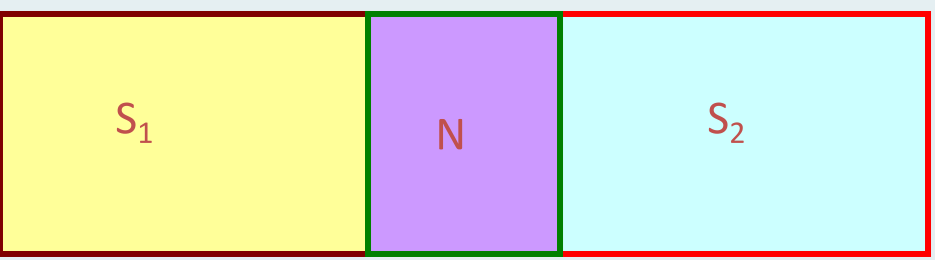
Josephson Effect



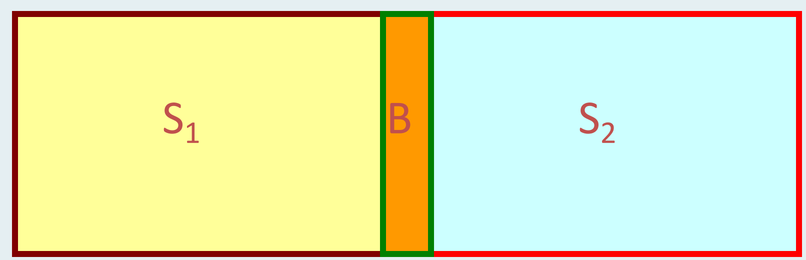
The ground state wavefunctions have different phases for S_1 and S_2 → $\psi_1 \sim e^{i\phi_1}$, $\psi_2 \sim e^{i\phi_2}$

Thus one might expect a current between them: DC Josephson Effect → $j \sim \text{Im}[\psi_1^* \psi_2] \sim \sin(\phi_2 - \phi_1)$

Experiments: Josephson junctions [Likharev, RMP 1979]

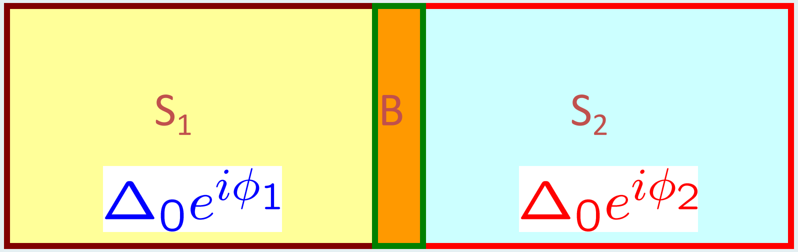


S-N-S junctions or weak links



S-B-S or tunnel junctions

Josephson effect in conventional tunnel junctions



Formation of localized subgap Andreev bound states at the barrier with energy dispersion which depends on the phase difference of the superconductors.

$$E_\pm = \pm\Delta_0\sqrt{1 - T\sin^2(\phi/2)},$$
$$T = 4/(4+Z^2),$$
Z is the dimensionless barrier strength.

The primary contribution to Josephson current comes from these bound states.

$$I = \frac{2e}{\hbar}\sum_{n=\pm}\sum_{k_\parallel}\frac{\partial E_n}{\partial\phi} f\left(E_n/k_B T_0\right)$$

Kulik-Omelyanchuk limit:

$T \to 1 \quad I(T_0 = 0) \sim |\sin(\phi/2)|$

$I_c R_N = \pi\Delta_0/e$

Ambegaokar-Baratoff limit:

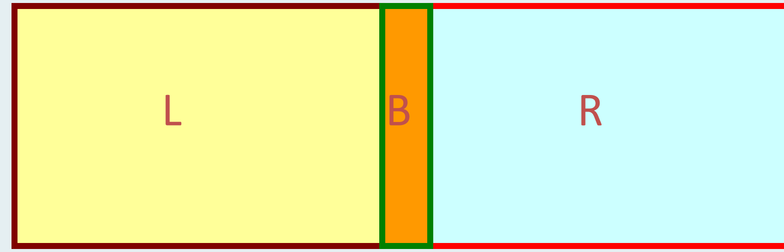
$T \to 0 \quad I(T_0 = 0) \sim T\sin(\phi)$

$I_c R_N = \pi\Delta_0/2e$

Both I_c and $I_c R_N$ monotonically decrease as we go from KO to AB limit.

Andreev bound states in Josephson junctions

Consider two p-wave superconductors Separated by a barrier modeled by a local potential of strength U_0 forming a Josephson tunnel unction



$$\hat{\Delta}_{\sigma k_y}(x,\hat{k}_x) \;=\; \begin{cases} \sigma\Delta_\beta, & s\text{-wave}, \\ \Delta_\beta\,\hat{k}_x/k_F, & p_x\text{-wave}, \end{cases}$$

b= R,L and s denotes spin

The superconductors acquire a phase difference f across the junction

$$\Delta_R = \Delta_0 e^{\imath\phi}, \qquad \Delta_L = \Delta_0 \;.$$

Solve the BdG equation across the junction with the boundary condition and find the subgap localized Andreev bound states

$$\begin{pmatrix} \varepsilon_{k_y}(\hat{k}_x) & \hat{\Delta}_{\sigma k_y}(x,\hat{k}_x) \\ \hat{\Delta}^\dagger_{\sigma k_y}(x,\hat{k}_x) & -\varepsilon_{k_y}(\hat{k}_x) \end{pmatrix} \psi_n = E_n \psi_n$$

$$\psi_L = \psi_R, \quad \partial_x\psi_R - \partial_x\psi_L = k_F Z\,\psi(0),$$
$$Z = 2mU_0/\hbar^2 k_F, \quad D = 4/(Z^2+4),$$

Solution for the Andreev states

On each side try a solution which is a superposition of right and left moving quasiparticles (index a denotes + or – for right or left movers) with momenta close to k_F

$$\psi_{\beta\sigma} = e^{\beta\kappa x}\left[A_\beta \begin{pmatrix} u_{\beta\sigma+} \\ v_{\beta\sigma+} \end{pmatrix} e^{i\tilde{k}_F x} + B_\beta \begin{pmatrix} u_{\beta\sigma-} \\ v_{\beta\sigma-} \end{pmatrix} e^{-i\tilde{k}_F x}\right]$$

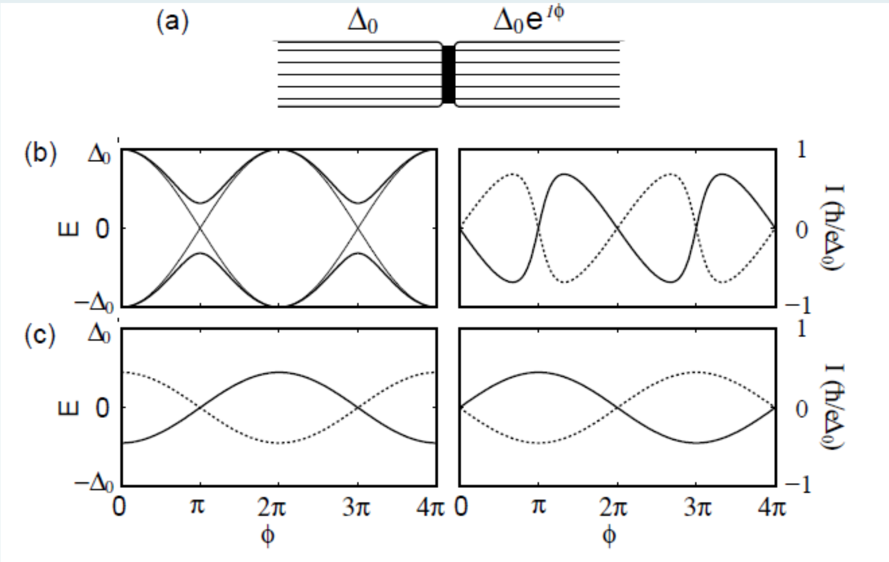
$$\eta_{\beta\sigma\alpha} = \frac{v_{\beta\sigma\alpha}}{u_{\beta\sigma\alpha}} = \frac{E + i\alpha\beta\hbar\kappa v_F}{\Delta_{\beta\sigma\alpha}}, \quad \kappa = \frac{\sqrt{\Delta_0^2 - |E|^2}}{\hbar v_F}$$

Substitute expressions for v and u in the boundary condition and demand non-zero solutions for A_b and B_b

$$\frac{(\eta_{-\sigma-} - \eta_{+\sigma-})(\eta_{-\sigma+} - \eta_{+\sigma+})}{(\eta_{-\sigma+} - \eta_{+\sigma-})(\eta_{-\sigma-} - \eta_{+\sigma+})} = 1 - D.$$

Leads to 4p periodic Josephson Current for p-p junctions

$$E_0^{(s)}(\phi) = -\Delta_0\sqrt{1 - D\sin^2(\phi/2)}, \quad s\text{-}s \text{ junction}$$
$$E_0^{(p)}(\phi) = -\Delta_0\sqrt{D}\cos(\phi/2), \quad p_x\text{-}p_x \text{ junction.}$$



Fractional AC Josephson effect

$$I_p(t) = \frac{\sqrt{D}e\Delta_0}{\hbar}\sin\left(\frac{\phi(t)}{2}\right) = \frac{\sqrt{D}e\Delta_0}{\hbar}\sin\left(\frac{eVt}{\hbar}\right)$$

Tunneling Hamiltonian approach

Consider two uncoupled 1D superconductors (corresponds to D=0) with two midgap states for each transverse momenta

$$v_{L0} = i u_{L0}^{*}, \qquad v_{R0} = -i u_{R0}^{*}.$$

$$\hat{\gamma}_{L0\sigma k_y}^{\dagger} = i \hat{\gamma}_{L0\bar{\sigma}\bar{k}_y}, \quad \hat{\gamma}_{R0\sigma k_y}^{\dagger} = -i \hat{\gamma}_{R0\bar{\sigma}\bar{k}_y}.$$

Thus the projection of the electron operator on the midgap state is given by

$$\mathcal{P}\hat{c}_{\sigma k_y}(x) = u_0(x)\hat{\gamma}_{0\sigma k_y} = v_0^{*}(x)\hat{\gamma}_{0\bar{\sigma}\bar{k}_y}^{\dagger}.$$

Now consider turning on a tunneling Hamiltonian between the left and the right superconductor

$$\hat{H}_{\tau} = \tau \sum_{\sigma k_y} [\hat{c}_{L\sigma k_y}^{\dagger}(\bar{l})\, \hat{c}_{R\sigma k_y}(l) + \hat{c}_{R\sigma k_y}^{\dagger}(l)\, \hat{c}_{L\sigma k_y}(\bar{l})].$$

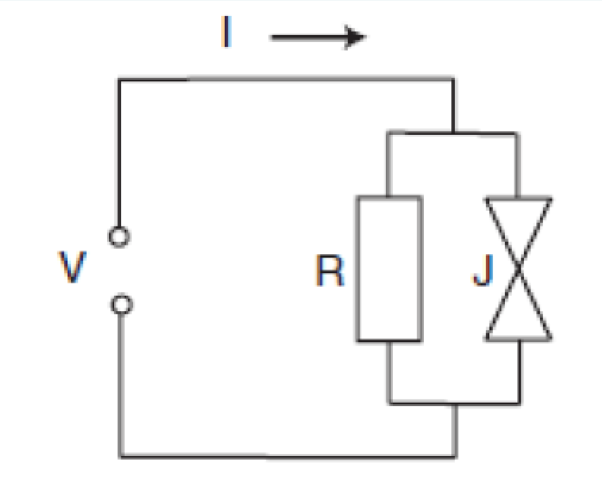
A little bit of algebra yields the Effective tunneling Hamiltonian For the subgap states

$$\mathcal{P}\hat{H}_{\tau} = \tau\, [u_{L0}^{*}(\bar{l}) u_{R0}(l) + \text{c.c.}]\, (\hat{\gamma}_{L0\uparrow}^{\dagger}\hat{\gamma}_{R0\uparrow} + \text{H.c.})$$
$$= \Delta_0 \sqrt{D}\, \cos(\phi/2)\, (\hat{\gamma}_{L0\uparrow}^{\dagger}\hat{\gamma}_{R0\uparrow} + \hat{\gamma}_{R0\uparrow}^{\dagger}\hat{\gamma}_{L0\uparrow}), \qquad ($$

The tunneling matrix elements vanish at f=p *where the states cross*

Consider a Josephson junction driven by a AC voltage (or irradiated by microwave frequency

$$V = V_0 + V_1 \cos(\omega t)$$
$$\phi = \phi_0 + \omega_J t + (2eV_1/\hbar\omega)\sin(\omega t)$$

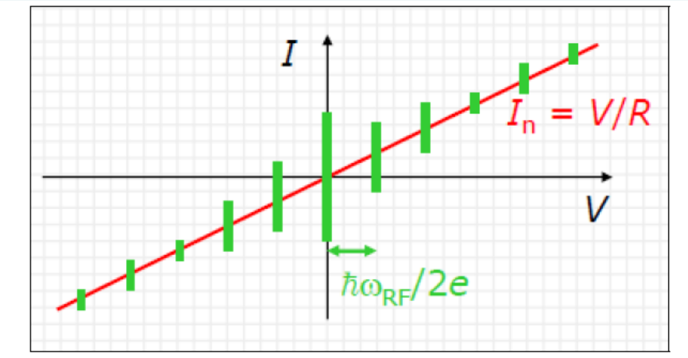


The resultant current in the circuit with a resistance R for a standard Josephson junction is

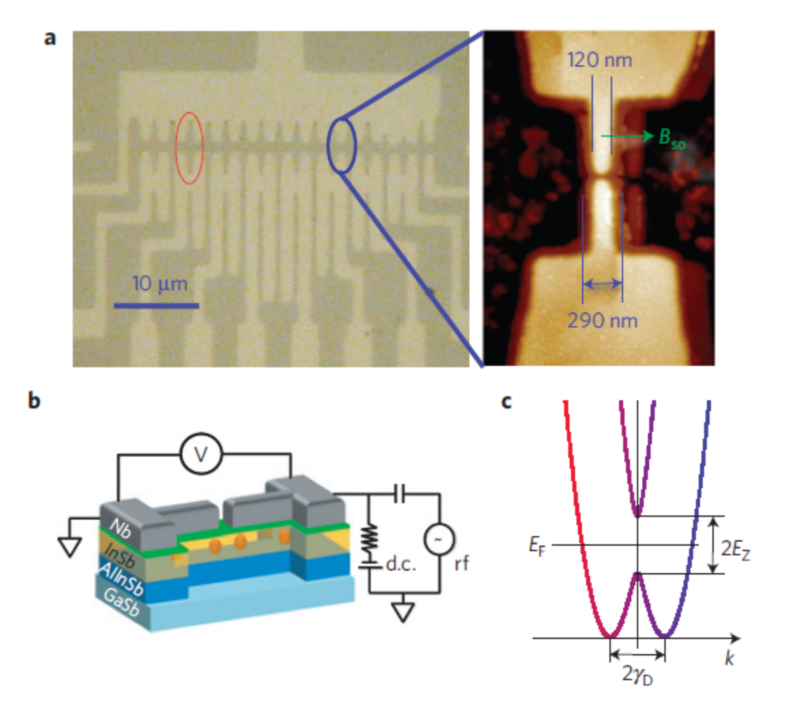
$$I = I_s + \frac{V_0}{R} = I_c \sum_{k=-\infty}^{\infty} (-1)^k J_k(\frac{2eV_1}{\hbar\omega}) sin(\phi_0 + \frac{2e}{\hbar} V_0 t - k\omega t) + \frac{V_0}{R}$$

Additional DC component in the current voltage charcteristics in the form of steps/spikes when

$$V_0 = \frac{k\hbar\omega}{2e}, \quad k = 0, \pm 1, \pm 2, \ldots$$



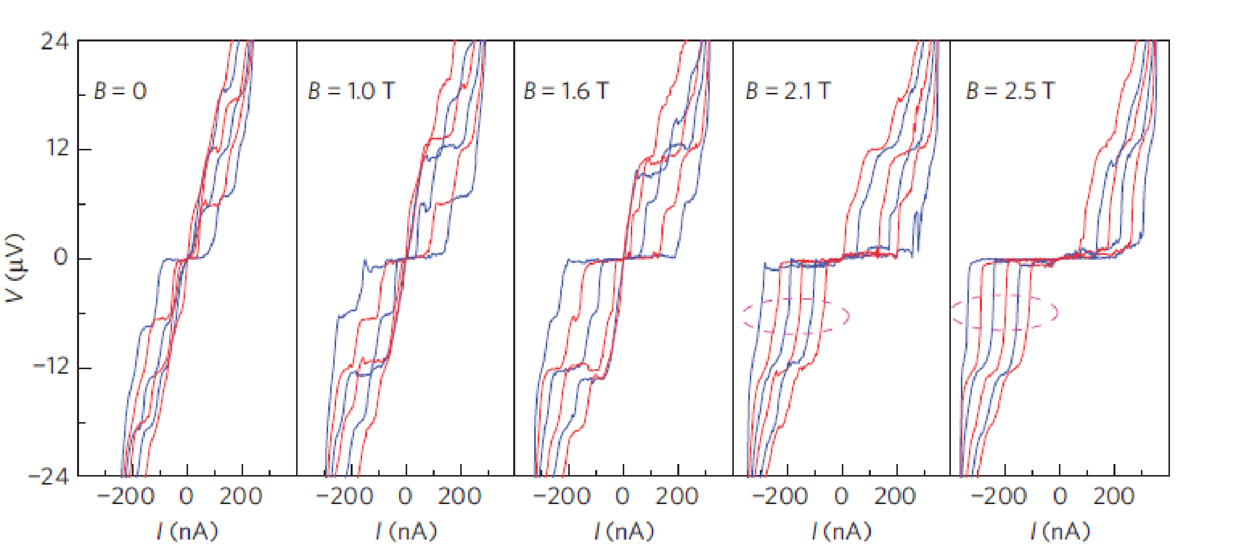
Recent experiments on doubling of Shapiro steps



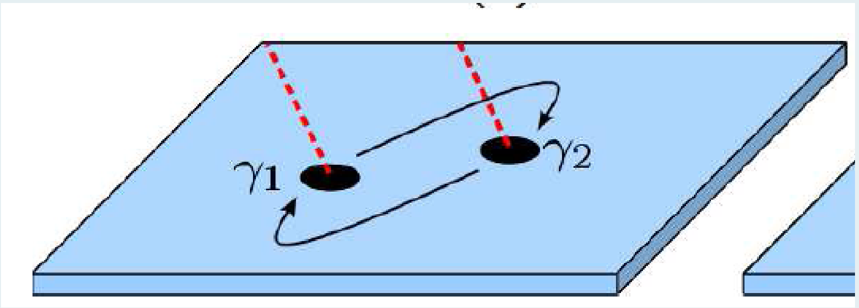
Recent experiments in 1D Semconductor wires with proximity induced superconductivity

Doubling of first Shapiro step from hn/2e to hn/e for B > 2 T.

Rokhinson et al Nat. Phys (2012)



Non-Abelian Statistics



Consider two vortices each of which has a zero energy Majorana fermion at the core → *Can occur at of p-wave superconductors (Ivanov 03)*

If one exchanges these two vortices as shown, the first vortex crosses the branch cut and gets a 2p phase. → *This is equivalent to 2p phase picked up by a Cooper pair and hence a phase p picked up by a individual fermion*

This exchange operation leads to a phase change of one of the two Majorana fermions →

$$\gamma_1 \to -\gamma_2, \quad \gamma_2 \to +\gamma_1.$$

This operation can be encoded by a Braid operator

$$\gamma_i \to B_{12}\gamma_i B_{12}^{\dagger}, \qquad B_{12} = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2).$$

It can be shown that if one applies this exchange on three vortices, the order of the exchange matters

$$[B_{i-1,i}, B_{i,i+1}] = \gamma_{i-1}\gamma_{i+1}.$$

Thus these vortices have non-Abelian exchange statistics

Conclusion

1. ***The fermion fractionalization in 1+1 D field theory has found a new avatar in condensed matter system.***

2. ***Out of the possible platforms for such fractionalization, the most interesting ones (experimentally) are superconducting nanowires***

3. ***Signature of the Majorana occur in midgap peak (?) and fractional Josephson effect.***

4. ***These particles obey anyonic statistics and one can construct universal quantum gates using them.***