Combinatorics and Graph Theory
NISER-AM Semester 4 of 2009

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1. **Elementary Enumeration [8, Chapter 1]**

   (a) Distinguishable and Indistinguishable Objects

      i. How many ways are there to pick 5 apples from 6 apples ? (Ans. 1)

      ii. How many ways are there to pick 5 boys from 6 boys ? (Ans. 6)

      iii. How many ways are there to pick 1 boy from 6 boys ? (Selecting r objects from n distinguishable objects automatically selects $n - r$ objects)

      iv. How many ways are there to pick 1 student from 3 boys and 2 girls? (Ans. 5)

      v. How many ways are there to pick 1 fruit from 3 apples and 2 oranges? (Ans. 2)

      vi. How many ways are there to pick 2 letters from 3 B's and 2 G's? (Ans. 3)

      vii. How many ways are there to pick 2 students from 3 boys and 2 girls? (Ans. 10)

   (b) Permutations and Combinations of the Objects

      i. How many ways are there to pick 1 hockey player and 1 football player from 4 hockey players and 5 football players ? (Ans. 4×5)

      ii. How many ways are there to make a 2-letter word if the letters are different ? (Ans. 26×25)

      iii. **Observation.** *The Multiplication Principle :* If one thing is done in m ways and a second thing is done in n ways independent of how the first thing is done, then the 2 things can be done in mn ways.

      iv. Permutation, r-permutation.

      v. How many ways can a pair of dice fall ? (Ans. 21 (indistingushable dice), 36 (distinguishable dice))

      vi. How many ways are there to arrange the letters BABA, BANANA? (Ans. $4!/(2!2!, 6!/(1!3!2!))$) Why so ?(Use different colored letters)

      vii. Permutation of multisets.

      viii. How many ways can we select 4 persons from 6 persons? (Ans. 15)

      ix. How many ways can we select r objects from n distinguishable objects when $n \geq r$? (Ans. $\binom{n}{r}$)

      x. r-combination, how combination and permutation differ, then find out the expression for $\binom{n}{r}$. (How many ways we select a final team from the 15 selected players for the upcoming Newzland series and how many ways we can select a batting order for the first test ?)

      xi. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$? (Inclusion and exclusion of Rahul Dravid in the final 11)

   (c) The Round Table

      i. Our convention for the counting the number of seating at a round table is that the seatings $s1$ and $s2$ are considered same iff everyone at the table has the same right/left hand neighbor in $s1$ and $s2$ (i.e., all rotations of a seating is considered as same seating).

ii. How many ways can all 6 Math faculties can be seated in a round table meeting ? (Ans. 5!, fix Prof. P.C. Das then ...)

iii. **Observation.** The number of ways of seating n persons in a round table is $(n-1)!$

iv. How many ways can 5 couples can be seated in a row (and at a round table) such that each couple seat together ? (Ans. $2!^5 \times 5!(2!^5 \times 4!)$)

v. How many ways can 5 men and 7 women can be seated in a row (and at a round table) such that no 2 men next to each other ? (Ans. $7! \times \binom{8}{5} \times 5!(6! \times \binom{7}{5} \times 5!)$)

(d) n Choose r with Repetition

i. What can we answer now for choosing r objects from n distinguishable objects with?

A. repetition not allowed and order matters; (Ans. ${}^nP_r$)

B. repetition allowed and order matters;(Ans. n^r)

C. repetition not allowed and order does not matter; (Ans. $\binom{n}{r}$)

D. repetition allowed and order does not matter; (Ans. ?)

ii. How many sequences are there consisting of 3 0/+'s and 7 1's ? (Ans. 10!/(3!7!))

iii. How many non-negative solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 7?$$

iv. How many ways put 7 indistinguishable balls into 4 boxes in a row (distinguishable)?

v. How many ways one can choose, with repetition allowed, 7 objects from 4 distinguishable objects ?

vi. How many sequences are there consisting of $n-1$ 0/+'s and r 1's ? (Ans. $(n-1+r)!/((n-1)!r!)$)

vii. How many non-negative solutions are there to the equation

$$x_1 + \ldots + x_n = r?$$

viii. How many ways put r indistinguishable balls into n boxes in a row (distinguishable)?

ix. How many ways one can choose, with repetition allowed, r objects from n distinguishable objects ?

x. **Observation.** With repetition allowed, the number of ways to choose r objects from n distinguishable objects is $\binom{n+r-1}{r}$.

xi. Picking with replacement (picking 5 cards from a deck with replcement) or Picking with repetition (picking 10 icecream cones from 5 flavors).

(e) Some More

i. Counting the number of strings of length n over english alphabets (with some restrictions like putting first letter A or B).

ii. Counting the number of polyndromes of length n over english alphabet. (Ans. $26^{\lceil n/2 \rceil}$)

iii. Counting the number of functions from a finite set A to finite set B (one-one functions, bijective functions, onto functions ?). (Ans. ${}^{|B|}P_{|A|}, |A|!, ?$)

iv. How many possibilities are there for 8 non-attacking rooks (distingushible, or 1 red, 2 black, 2 white, 3 blue) on an 8×8 chessboard, where a rook can attack by vertically and horizontally ? [3, Page 68] (Ans. $8!(8! \times 8!, 8! \times 8!/(1!2!2!3!))$)

v. How many rectangles and squares are there in $n \times n$ chessboard ? (Ans $\binom{n+1}{2} \times \binom{n+1}{2}, 1^2 + 2^2 + \ldots + n^2$)

vi. A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated ?

2. **Pigeonhole (Dirichilet) Principle**

   (a) In a group of 367 people, two people must have same birth day.

   (b) There are two person in Bhubaneswar and Cuttack having same number of hairs in their body. (Population: 9 millions and # hairs per person $\leq$ 7 millions)

   (c) If one collects 10 points from an equilateral triangle of sides having length 1 unit, then there must be 2 points having distance atmost $\frac{1}{3}$ unit.

   (d) Pigeonhole principle.

   (e) Let acquaintance relation is symmetric (i.e., A knows B iff B knows A). In a group of $50(n > 1)$ people, two people must have same number of acquaintances [6].

   (f) One of $101(n+1)$ numbers from the set $\{1, 2, \ldots, 200(2n)\}$ is divisible by another? (Hint: write each number as $2^s a$, where a is odd)

   (g) A Chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games in total. There must be a sequence of successive days on which he plays 21 games [6].

   (h) **(Erdoš and Szekeres)** Given a sequence of $n^2+1(pq+1)$ distinct integers, either there is an increasing subsequence of $n+1(p+1)$ terms or a decreasing subsequence of $n+1(q+1)$ terms [6, 9].

   (i) Ramsey number/theory.

3. **Principle of Inclusion and Exlusion (PIE)**

   (a) In a class of NISER 1st semester, the students failed only in Mathematics and Physics subjects. If the total number of students failed is 7, number of studens failed in Mathematics is 5 and number of students failed in Physics is 3, then how many students failed in both Physics and Mathematics ?

   (b) Principle of Inclusion and Exclusion (Using set and using the property satisfied by elements of sets [9]).

   (c) How many 4-letter words begin or end with a vowel? ((without)using PIE)

   (d) How many integers between 1 and 1000 are (i) not divisible by either 2 or 5 (ii) not divisible by $2, 5$ or 11 ?

   (e) Proving Euler's ϕ function [9, Page 410].

   (f) Derangements [8, Page 32];
   The Hatcheck Problem: How many ways can a hatcheck girl hand back the n hats of n gentlemen, 1 to each gentleman, with no man getting his own hat ?

4. **Counting Through Recurrence Relation [9, 8]**

   (a) Simple/compound interest, Story of Chess inventor, Fibonacci Sequence;

   (b) A recurrence relation for a sequence $\{a_i\}, i \geq 0$ or 1 that defines a_n in terms of $a_0, a_1, \ldots, a_{n-1}$ and $n > k$ for some particular integer k, with the terms $a_0, \ldots, a_k$ called initial/boundary condition.

   (c) How many regions do n straight lines (non-parallel and no 3 lines intersect at the same point) divide the plane ? (Ans. $a_n = a_{n-1} + (n+1), a_1 = 2$)

   (d) Counting the number of decimal strings of length $n+1$ which contain even number of 0s. (Ans: $C_{n+1} = 9C_n + (10^n - C_n) = 8C_n + 10^n, C_1 = 9$)

(e) Counting the number of binary strings of length n which do not contain any consecutive 0s. (Ans: $C_n = C_{n-1} + C_{n-2}, C_1 = 2, C_2 = 3$)

(f) Tower of Hanoi/Brahma.

(g) Counting number of de-arrangements of n objects. (Ans: $D_n = (n-1)(D_{n-1} + D_{n-2})$)

(h) Counting the number of different partitions of $n+1$-element set (i.e., Bell numbers). (Ans: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k$)

5. **Putting r balls into n boxes [9, Page 51], [8, Page 35]**

(a) Distinguishable and In-distinguishable objects.

(b) Number of ways can r-distinguishable balls be put into n-distinguishable boxes. (e.g., Count the number of functions from A to B) (Ans: n^r)

(c) Item 5b with condition that no box is empty (e.g., count the number of onto functions from A to B) (Ans: $\sum_{k=0}^{n}(-1)^k \binom{n}{r}(n-k)^r$)

(d) Number of ways can r-distinguishable balls be put into n-indistinguishable boxes with no box is empty. (This number is denoted by $\left\{ \begin{matrix} r \\ n \end{matrix} \right\}$ or $S(r,n)$ and is called as Stirling number of the second kind. Number of ways to partition a set of r things into n non-empty subsets. $B_r = \sum_{i=1}^{r} \left\{ \begin{matrix} r \\ i \end{matrix} \right\}$) (Ans: $\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \binom{n}{r}(n-k)^r$ or, $\left\{ \begin{matrix} r \\ n \end{matrix} \right\} = \left\{ \begin{matrix} r-1 \\ n-1 \end{matrix} \right\} + n \left\{ \begin{matrix} r-1 \\ n \end{matrix} \right\}, 1 < n < r, \left\{ \begin{matrix} r \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} r \\ r \end{matrix} \right\} = 1$)

(e) Number of ways can r-distinguishable balls be put into n-indistinguishable boxes.(Number of ways to partition a set of r things into at most subsets.) (Ans. $\sum_{i=1}^{n} \left\{ \begin{matrix} r \\ i \end{matrix} \right\}$)

(f) Number of ways can r-indistinguishable balls be put into n-distinguishable boxes. (Ans: $\binom{n+r-1}{r}$ Item 1(d)viii)

(g) Item 5f with condition that no box is empty (Ans: $\binom{r-1}{r-n} = \binom{r-1}{n-1}$ Item 1(d)viii)

(h) Number of ways can r-indistinguishable balls be put into n-indistinguishable boxes with no box is empty. (Partitioning an integer $r > 0$ into n parts) (Ans: $\Pi(r,n) = \Pi(r-1,n-1) + \Pi(r-n,n), \Pi(r,n) > 0$ if $n > r$, $\Pi(r,r) = \Pi(r,1) = 1$)

(i) Number of ways can r-indistinguishable balls be put into n-indistinguishable boxes. (Partitioning an integer $r > 0$ into atmost n parts) (Ans: $\Pi(r+n,n)$)

6. **Generating Function [9, 8]**

(a) Power Series: a power series is an infinite series of the form $\sum_{i=0}^{\infty} a_k x^k$. Radius of Convergence.

(b) Taylor series expansion: Suppose $f(x)$ is a function having darivatives of all orders for all x in an interval containing 0.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

(c) Some useful expansions

i. $\dfrac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, for $|x| < 1$;

ii. $e^x = \sum_{i=0}^{\infty} \dfrac{1}{i!}x^i$, for $|x| < \infty$;

iii. $sinx = \sum_{i=0}^{\infty}(-1)^k \frac{1}{(2i+1)!}x^{2i+1}$, for $|x| < \infty$;

iv. $\log(1+x) = \sum_{i=1}^{\infty}(-1)^{k+1}\frac{1}{i}x^i$, for $|x| < 1$.

(d) Generating Function: The generating function for the sequence $(a_k), k \geq 0$ is defined to be

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x^1 + a_2 x^2 + \dots$$

(e) Some Examples: Find the generating function of

i. $a_k = \binom{n}{k}$;

ii. $a_k = \frac{1}{(k+2)!}$;

(f) Some Examples: Find the sequence of the generating function

i. $f(x) = \frac{x^2}{1-x}$;

ii. $f(x) = cosx$;

iii. $f(x) = \frac{1}{(1+x)^2}$.

(g) Operating on generating functions: Let $A(x), B(x)$ and $C(x)$ be the generating functions for the sequences $(a_k), (b_k)$ and (c_k) respectively. If

i. $A(x) = B(x) + C(x)$, then $a_k = b_k + c_k$;

ii. $A(x) = B(x)C(x)$, then $a_k = \sum_{i=0}^{k} b_i c_{k-i}$. The sequence (a_k) is called convolution of the two sequences (b_k) and (c_k).

iii. Find the sequence of $f(x) = \frac{1}{(1+x)^2}$;

iv. Find the sequence of $f(x) = \frac{1+x+x^2+x^3}{1-x}$.

(h) Examples:

i. How many ways are there to distribute 10 balls to 3 persons ?

ii. How many ways are there to distribute 10 balls to 3 persons such that each person can get at least 2 balls [and at most 5 balls]?

iii. How many ways are there to select 25 balls from unlimited supplies of red, white and blue balls such that at least 3 red balls and at most 5 white balls are selected ?

iv. How many ways one can make 2 dollars using pannies, nickels and dimes ?

v. How many non-negative/positive solutions are there of $x_1 + 7x_2 + 3x_3 = 100$?

(i) Exponential generating function [9]

i. Exponential generating function for the sequence $(a_k), k \geq 0$ is defined to be

$$H(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \dots$$

ii. $H(x)$ for the sequence $(a_k = 1)$;

iii. $H(x)$ for the sequence $(a_k = \alpha^k)$;

iv. How many words of length at most 5 are there using there letters say, a, b and c (of atmost one a, at most one b and atmost 3 cs).
How many multi-subsets of size at most 5 are there using there letters say, a, b and c (of atmost one a, at most one b and atmost 3 cs).

v. Number of distingushable permutations of length k using p types of objects with upto n_i objects from ith object.

vi. Number of ways to put n distinguishable balls into k distinguishable cells. Compare with the result we found earlier. [9, page 325].

7. **Solving Reccurence Relations [9, 3]**

   (a) Linear homogeneous reccurence relation:

      i. Linear homogeneous reccurence relation: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_p a_{n-p}$, $n \geq p$, where $c_1, \ldots, c_p$ are constants and $c_p \neq 0$.

      ii. In general, we the initial conditions are disregarded, a recurrence relation has many solutions.

      iii. Theorem [3, 9]: If q is root of the polynomial

$$x^p - c_1 x^{p-1} - c_2 x^{p-2} - \cdots - c_p = 0$$

      iff $a_n = q^n$ is a solution of the linear homogeneous recurrence relation

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_p a_{n-p} = 0, n \geq p, c_p \neq 0$$

      iv. $x^p - c_1 x^{p-1} - c_2 x^{p-2} - \cdots - c_p = 0$ is called the *characteristic equation* of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_p a_{n-p}$, $n \geq p$, where $c_1, \ldots, c_p$ are constants and $c_p \neq 0$.

      v. Theorem [3, 9]: If the characteristic polynomial equation has p distinct roots $q_1, q_2, \ldots, q_p$, for each initial condition there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $\lambda_1 q_1 + \lambda_2 q_2 + \ldots + \lambda_p q_p$ is the solution.

      vi. Solve $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}, n \leq 3$ subject to the initial value $h_0 = 1, h_1 = 2, h_2 = 0$.

      vii. Compute F_k, the kth Fibonnaci number;

      viii. Compute number of ways a string containing '$*$' (laghu), '$-$' (guru) and ' ' (space) such that no string contains two consecutive ' 's (spaces).

      ix. Let $q_1, \ldots, q_t$ be the distinct roots of the of the characteristic equation $x^p - c_1 x^{p-1} - c_2 x^{p-2} - \cdots - c_p = 0$ with multiplicity $s_1, \ldots, s_t$, then the solutions of the corresponding reccurence solution is

$$\sum_{i=1}^{t} (\lambda_{i,1} + n\lambda_{i,2} + \ldots + n^{s_i - 1} \lambda_{i,s_i}) q_i^n$$

      x. Solve the reccurence $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$, $a_1 = 2$.

      xi. Solve the reccurence $a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4}$ with $a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2$.

   (b) Using Generating Functions

      i. Solve $a_{n+1} = 2a_n$ with $a_0 = \frac{1}{2}$

      ii. Legitmate codewords: How many codewords of length k from the alphabet $\{0, 1, 2, 3\}$ having an even number of 0's are there ? ($a_{k+1} = a_k + 4^k$, $a_1 = 3$)

      iii. Solve Derangement Problem;

      iv. Simultaneous Equations for generating functions [9, Section 6.1.4, 6.3.3]

8. **The Polya Theory of Counting [9]**

   (a) Group action [8];

   (b) Permutation group;

   (c) Equivalence class by the action of permutation group;

   (d) Orbit, invariant under a permutation($inv(\pi)$), stabilizer($st(a)$);

   (e) Lemma: Suppose that G is a group of permutations on a set A and a is in A. Then $|st(a)|.|C(a)| = |G|$.

(f) Burnside's lemma: Let G be a group of permuatations of a set A and let S be the equivalence relation on A induced by G. Then the number of equivalence classes in S is given by $\frac{1}{|G|}\sum_{\pi\in G} inv(\pi)$.

(g) Equivalent coloring problem;

(h) Cycle decomposition of permutation;

(i) A special case of Polya's theorem: Suppose that G is a permutation group of the set A and $C(A,R)$ is the set of colorings of elements of A using colors in R, a set of m elements. Then the number of didtinct colorings in $C(A,R)$ is given by

$$\frac{1}{|G|}[m^{cyc(\pi_1)} + m^{cyc(\pi_2)} + \ldots + m^{cyc(\pi_k)}]$$

where $G = \{\pi_1, \ldots, \pi_k\}$.

(j) Number of colorings of 2×2 square by 2 colors which are equivalent under rotation by multiples of 90^0.

(k) Number of necklaces of 4 beads by 3 colors which are equivalent under rotations and flips.

(l) Number of rotation symmetric Boolean functions on n variables.

(m) Number of dihedral group invariant Boolean functions on n variables.

(n) Polya's theorem

i. Cycle index of a permutation group;

ii. The inventory of colorings(weight of colors, colorings, inventory of a set of colorings);

iii. If colorings f and g are equivalent, they have the same weight;

iv. Examples of inventory of colorings;

v. Polya's Theorem: Suppose that G is a group of permutations on a set D and $C(D,R)$ is the collection of all colorings of D using colors in R. If w is a weight assignment on R, the pattern inventory of colorings in $C(D,R)$ is given by

$$P_G(\sum_{r\in R} w(r), \sum_{r\in R}[w(r)]^2, \ldots, \sum_{r\in R}[w(r)]^k)$$

where $P_G(x_1, x_2, \ldots, x_k)$ is the cycle index.

vi. Lemma: Suppose that D is divided up into disjoint sets $D_1, D_2, \ldots, D_p$. Let C be the subset of $C(D,R)$ that consists of all colorings f with the property that if $a, b \in D_i$, some i, then $f(a) = f(b)$. Then the inventory of the set C is given by

$$[w(1)^{|D_1|} + w(2)^{|D_1|} + \ldots + w(m)^{|D_1|}] \times [w(1)^{|D_2|} + w(2)^{|D_2|} + \ldots + w(m)^{|D_2|}]\times$$
$$\ldots \times [w(1)^{|D_p|} + w(2)^{|D_p|} + \ldots + w(m)^{|D_p|}].$$

vii. Lemma: Suppose that $G^* = \{\pi_1^*, \pi_2^*, \ldots, \pi_k^*\}$ is a group of permutations of $C(D,R)$. For each $\pi^* \in G^*$, let $\overline{w}(\pi^*)$ be the sum of the weights of all colorings f in $C(D,R)$ left invariant by π^*. Suppose that $C_1, C_2, \ldots, C_t$ are the equivalence classes of colorings and $w(C_i)$ is the common weight of all f in C_i. Then

$$w(C_1) + w(C_2) + \ldots + w(C_t) = \frac{1}{|G^*|}[\overline{w}(\pi_1^*) + \overline{w}(\pi_2^*) + \ldots + \overline{w}(\pi_k^*)].$$

viii. Proof of Polya's theorem.

9. **Systems of Distinct Representatives [1, 10]**
    (a) Example and Definition of SDR [1];
    (b) Hall's theorem [10];
    (c) Lower bound of the number of SDRs [10];
    (d) Latin square and Latin rectangle [1];
    (e) Construction of Latin squares from Latin rectangles [10];
    (f) Lowerbound of the count of Latin squares of order n [10];

10. **Introduction to Graph Theory [2]**
    (a) Basic notations and definitions
        i. Why graph theory ?
        ii. incident of edge with vertex, adjacent, neighbours ($N_G(v)$), loop, parallel edge;
        iii. finite graph, infinite graph, null graph, simple graph;
        iv. complete graph(K_n), bipartite graph(bipartition), complete bipartite graph($K_{m,n}$), star, path(P_n), cycle(C_n), length of path/cycle, connected graph, planar graph;
        v. incidence matrix, adjacency matrix, bipartite adjacency matrix;
        vi. degree of vertex ($d(v)$), $\delta(G), \Delta(G), d(G)$, k-regular graph.
        vii. For any graph G, $\sum_{v \in V} d(v)$ is even. The number of vertices of odd degree is even.
        viii. isomorphism, testing isomorphism, self complementary, automorphism, exercises 1.2.7, 1.2.9, 1.2.10, 1.2.11, 1.2.13, 1.2.16 from [2].
        ix. Disjoint graph, edge-disjoint graph, union of graphs($\cup$), disjoint union of graphs($+$), connected component, intersection of graphs($\cap$), cartesian product of graphs ($\Box$), $P_n \Box P_m, C_n \Box P_2, C_n \Box P_m$.
        x. Directed graph: definition, incidence function, head, tail, dominates, in-neighbours ($N_D^-(v)$, out-neighbours ($N_D^+(v)$, indegree($d_D^-(v)$) outdegree($d_D^+(v)$), strict digraph, underlying graph of D ($G(D)$), associated digraph of G ($D(G)$), orientation of G, tournament, k-diregular graph, source, sink, isomorphism between 2 digraph, incidence matrix & adjacency matrix of digraph.
    (b) Subgraphs and Supergraphs
        i. subgraph($\subseteq$), supergraph($\supseteq$), edge deletion($G \setminus e$), vertex deletion ($G - v$), F-subgrapgh.
        ii. Theorem: Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.
        iii. acyclic graph, contrapositive of previus theorem, Exercises 2.1.2, 2.1.3 [2].
        iv. Maximal and Minimal: maximal and minimal graph, maximal path of a graph, circumference and girth of a graph.
        v. spanning subgraph $G \setminus S$, every simple graph is a spanning subgraph of a complete graph, spanning supergraph ($G + S$), join of two graphs ($G \vee H$), wheel with n spokes ($W_n = C_n \vee K_1$), spanning path and cycle.
        vi. induced subgraph ($G - X$), $G[Y]$, edge induced subgraph($G[S]$).
        vii. weighted graph, Travelling salesman problem.
        viii. Exercises: 2.2.1, 2.2.9, 2.2.10, 2.2.13 [2]
    (c) Modification of graphs
        i. identification of two non-adjacent vertices ($G/\{x, y\}$), contraction of an edge (G/e);
        ii. vertex splitting (# ?), subdividing edge;

iii. Decomposition of a graph (# ?), path decomposition, cycle decomposition, even graph;
iv. Theorem: A graph admits a cycle decomposition iff it is even.
v. exercise 2.4.5, 2.4.6, 2.4.8(b) [2]
vi. Covering of a graph, uniform covering (k-cover, 1-cover), path/cycle covering,

(d) Edge cuts and bonds

i. $E[X, Y]$, edge cut ($\partial(X)$), bipartite and connected graph in terms of edge cut, trivial cut ($\partial(v)$), $d(X)$.
ii. Theorem: For any graph G and any subset X of $V(G)$, $|\partial(X)| = \sum_{v \in X} d(v) - 2e(X)$.
iii. bond
iv. Theorem: In a connected graph G, a nonempty edge cut $\partial(X)$ is a bond if and only if both $G[X]$ and $G[V\ X]$ are connected.

(e) Connected graphsi [2, 4]

i. walk (uv-walk), u connects v, segment, closed walk,trail, path, connected graph, connection is an equivalence relation, components, distance between u and v ($d_G(u, v)$), arc (forward and reverse arc), cycle, k-cycle odd/even cycle.
ii. Let G be a nonempty graph with atleast two vertices. Then G is bipartite iff it has no odd cycles [4].
iii. Exercises 3.1.1, 3.1.2 [2]
iv. cut edge (bridge)
v. Th: An edge e of a graph G is a bridge iff e belongs to no cycle of G.
vi. Exercise 3.2.3 [2].
vii. Euler trail, konigsberg problem, Euler tour, eulerian graph
viii. A connected graph is eulerian iff it is even graph [4].
ix. Fleury's algorithm [4]
x. Directed walk/trail/path/cycle, strongly connected digraph, equivalence class, strong component, directed euler trail, eulerian digraph.
xi. A connected digraph is eulerian iff it is even.

(f) Tree

i. acyclic graph, tree, forest, non-isomorphic trees on 6 vertices .
ii. Prop: In a tree, any two vertices are connected by exactly one path.
iii. leaf of tree (there exists a vertex of degree atmost 1 (for nontrivial tree, it is exactly 1)).
iv. Props: Every non trivial tree has atleast 2 leaves.
v. Th: If T is tree, then $e(T) = v(T) - 1$.
vi. rooted tree, branching
vii. Exercises: 4.1.4, 4.1.3 [2]
viii. subtree, spanning tree,
ix. Prop: A graph is connected iff it has a spanning tree;
x. Th: A graph is bipartite iff it contains no odd cycle. So, either a graph is bipartite or contains a odd cycle.
xi. Cayley's theorem: The number of labelled trees on n-vertices is n^{n-2}. That is number spanning trees of a complete graph on n vertices ($t(K_n)$) is n^{n-2}.

(g) Cut vertex

i. Cut vertices;

ii. Theorem: v is a cut vertex of a graph G iff there are two vertices u and w of G, both different from v, such that v is on every uw-path in G [4].
iii. cut vertices of K_n, P_n.
iv. Let G be a graph with atleast 2 vertices. Then G has atleast 2 vertices which are not cut vertices. [4]
v. Theorem: A connected graph on 3 or more verices has no cut vertices iff any two distinct vertices are connected by two internally disjoint paths.
vi. Connectivity [4], Separation, separating vertex, nonseparable graph;
vii. Theorem: A connected graph is nonseparable iff any two of its edges lie on a common cycle.
viii. Proposition 5.3 [2].

(h) Planar graph

i. Plane graph, planar graph; curve, closed curve, Jordan curve, arcwise-connected; Jordan curve theorem, interior and exterior of a curve.
ii. $K_5, K_{3,3}$ are non-planar.
iii. Euler's formula
iv. All planar embeddings of a connected planar graph have the same number of edges.
v. Let G be a plane graph with n vertices, e edges, f faces and k connected components. Then $n - e + f = k + 1$;
vi. degree of a face ($d(f) \geq 3$ for any interior face f);
vii. Let G be a simple planar graph with n vertices and e edges, where $n \geq 3$. Then $e \leq 3n - 6$.
viii. There is a v in $V(G)$ with $d(v) \leq 5$ for a simple planar graph G.
ix. $K_5, K_{3,3}$ is non-planar.

(i) Vertex colouring [4, 2]

i. Colouring problem, k-colourable, chromatic number/index ($\chi(G)$);
ii. Theorem
   A. $\chi(G) \leq V(G)$;
   B. If $H \subseteq G$ then $\chi(H) \leq \chi(G)$;
   C. $\chi(K_n) = n, \chi(C_n) =?, \chi(S_n) =?$;
   D. If $K_n \subseteq G$ then $\chi(G) \geq n$;
   E. If $G_1, \ldots, G_n$ are components of G then $\chi(G) =?$;
   F. $\chi(G) = 2$ iff G is bipartite; $\chi(G) \geq 3$ iff G has an odd cycle.
iii. $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max\{d(v) : v \in V(G)\}$ is the maximum vertex degree of G.
iv. Brooks Theorem: If $\Delta(G) \geq 3$ and $G \neq K_n$ then $\chi(G) \leq \Delta(G)$.
v. Exercise 6.1 [4], coloring of petersen graph(odd cycle and $\Delta(G) \geq 3$).
vi. Sequential vertex coloring algorithm.

(j) Chromatic polynomial [5]

i. $P_n(\lambda) = \sum_{i=1}^{\lambda} c_i \binom{\lambda}{i}$, where c_i be the different ways of properly coloring G using i colors ($c_n = n!$).
ii. Example: fig 8.4 [5].
iii. A graph of n-vertices is a complete graph iff its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1) \ldots (\lambda - n + 1)$.
iv. Recursive construction [5, Theorem 8.6] with example [5, fig 8.5].
v. $P_n(\lambda)$ is the weighted sum of $P_k(\lambda)$ of K_k for $k \leq n$.

vi. Theorem [7]: Let G be a graph with n points, q edges and k components $G_1, G_2, \ldots, G_k$. Then

A. $P_n(\lambda)$ has degree n;

B. The coefficient of λ^n is 1;

C. The coefficient of λ^{n-1} is $-q$;

D. The constant term in $P_n(\lambda)$ is 0;

E. $P_n(\lambda) = \prod_{i=1}^{k} P_{n_i}(\lambda)$;

F. The smallest exponent of λ in $P_n(t)$ with nonzero coefficient is k.

vii. An n-vertex graph is a tree iff its chromatic polynomial $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$.

(k) Edge coloring [4]

i. Edge coloring and edge chromatic number($\chi_1(G)$);

ii. $H(i,j)$, Kempe chain;

iii. $\chi_1(G) \geq \Delta(G)$;

iv. Theorem: Let G be a nonempty bipartite graph. Then $\chi_1(G) = \Delta(G)$.

v. Equivalence between latin square and edge coloring of bipartite graph.

vi. Let $G = K_n$, the complete graph on n vertices, $n \geq 2$. Then

$$\chi_1(G) = \begin{cases} \Delta(G) + 1(\text{ i.e., } n) & \text{if } n \text{ is odd} \\ \Delta(G)(\text{ i.e., } n-1) & \text{if } n \text{ is even.} \end{cases}$$

(l) Hamiltonian graph [4]

i. Hamiltonian path, Hamiltonian cycle, Hamiltonian graph, Examples of figure 3.21 and 3.22, Maximal non-Hamiltonian graph.

ii. Theorem (Dirac): If G is a simple graph with $n(\geq 3)$ vertices and $d(v) \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian.

iii. Theorem: Let G be a simple graph with n-vertices and let u and v are non-adjacent in G such that

$$d(u) + d(v) \geq n.$$

Let $G + uv$ is Hamiltonian iff $G + uv$ is Hamiltonian.

iv. Theorem: A simple graph G is Hamiltonian iff its closure $c(G)$ is Hamiltonian.

v. Corollary: Let G be a simple graph on n vertices, with $n \geq 3$. If $c(G) = K_n$, then G is Hamiltonian.

vi. Travelling Salesman problem, Two-optimal algorithm.

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