Combinatorics and Graph Theory
NISER-MA401 Semester 4 of 2010

Instructor: Deepak Kumar Dalai

6th January 2010

1. **Elementary Enumeration [3, Chapter 1]**

   (a) Some basic counting

      i. Counting the number of strings of length n over english alphabets (with some restrictions like putting first letter A or B).

      ii. Counting the number of polyndromes of length n over english alphabet. (Ans. $26^{\lceil n/2 \rceil}$)

      iii. Counting the number of functions from a finite set A to finite set B (one-one functions, bijective functions, onto functions ?). (Ans. ${}^{|B|}P_{|A|}, |A|!, ?$)

      iv. How many possibilities are there for 8 non-attacking rooks (distingushible, or 1 red, 2 black, 2 white, 3 blue) on an 8×8 chessboard, where a rook can attack by vertically and horizontally ? [1, Page 68] (Ans. $8!(8! \times 8!, 8! \times 8!/(1!2!2!3!))$)

      v. How many rectangles and squares are there in $n \times n$ chessboard ? (Ans $\binom{n+1}{2} \times \binom{n+1}{2}, 1^2 + 2^2 + \ldots + n^2$)

      vi. A classroom has 2 rows of 8 seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated ?

   (b) Distinguishable and Indistinguishable Objects

      i. How many ways are there to pick 5 apples from 6 apples ? (Ans. 1)

      ii. How many ways are there to pick 5 boys from 6 boys ? (Ans. 6)

      iii. How many ways are there to pick 1 boy from 6 boys ? (Selecting r objects from n distinguishable objects automatically selects $n - r$ objects)

      iv. How many ways are there to pick 1 student from 3 boys and 2 girls? (Ans. 5)

      v. How many ways are there to pick 1 fruit from 3 apples and 2 oranges? (Ans. 2)

      vi. How many ways are there to pick 2 letters from 3 B's and 2 G's? (Ans. 3)

      vii. How many ways are there to pick 2 students from 3 boys and 2 girls? (Ans. 10)

   (c) Permutations and Combinations of the Objects

      i. How many ways are there to pick 1 hockey player and 1 football player from 4 hockey players and 5 football players ? (Ans. 4×5)

      ii. How many ways are there to make a 2-letter word if the letters are different ? (Ans. 26×25)

      iii. **Observation.** *The Multiplication Principle :* If one thing is done in m ways and a second thing is done in n ways independent of how the first thing is done, then the 2 things can be done in mn ways.

      iv. Permutation, r-permutation.

v. How many ways can a pair of dice fall ? (Ans. 21 (indistingushable dice), 36 (distinguishable dice))

vi. How many ways are there to arrange the letters BABA, BANANA? (Ans. $4!/(2!2!, 6!/(1!3!2!)))$ Why so ?(Use different colored letters)

vii. Permutation of multisets.

viii. How many ways can we select 4 persons from 6 persons? (Ans. 15)

ix. How many ways can we select r objects from n distinguishable objects when $n \geq r$? (Ans. $\binom{n}{r}$)

x. r-combination, how combination and permutation differ, then find out the expression for $\binom{n}{r}$. (How many ways we select a final team from the 15 selected players for the upcoming Newzland series and how many ways we can select a batting order for the first test ?)

xi. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$? (Inclusion and exclusion of Rahul Dravid in the final 11)

7th January 2010

(d) The Round Table

i. Our convention for the counting the number of seating at a round table is that the seatings $s1$ and $s2$ are considered same iff everyone at the table has the same right/left hand neighbor in $s1$ and $s2$ (i.e., all rotations of a seating is considered as same seating).

ii. How many ways can all 6 Math faculties can be seated in a round table meeting ? (Ans. 5!, fix Prof. P.C. Das then ...)

iii. **Observation.** The number of ways of seating n persons in a round table is $(n-1)!$

iv. How many ways can 5 couples can be seated in a row (and at a round table) such that each couple seat together ? (Ans. $2!^5 \times 5!(2!^5 \times 4!)$)

v. How many ways can 5 men and 7 women can be seated in a row (and at a round table) such that no 2 men next to each other ? (Ans. $7! \times \binom{8}{5} \times 5!(6! \times \binom{7}{5} \times 5!)$)

(e) n Choose r with Repetition

i. What can we answer now for choosing r objects from n distinguishable objects with?

A. repetition not allowed and order matters; (Ans. ${}^nP_r$)

B. repetition allowed and order matters;(Ans. n^r)

C. repetition not allowed and order does not matter; (Ans. $\binom{n}{r}$)

D. repetition allowed and order does not matter; (Ans. ?)

ii. How many sequences are there consisting of 3 0/+'s and 7 1's ? (Ans. $10!/(3!7!)$)

iii. How many non-negative solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 7?$$

iv. How many ways put 7 indistinguishable balls into 4 boxes in a row (distinguishable)?

v. How many ways one can choose, with repetition allowed, 7 objects from 4 distinguishable objects ?

vi. How many sequences are there consisting of $n-1$ 0/+'s and r 1's ? (Ans. $(n-1+r)!/((n-1)!r!)$)

vii. How many non-negative solutions are there to the equation

$$x_1 + \ldots + x_n = r?$$

viii. How many ways put r indistinguishable balls into n boxes in a row (distinguishable)?

ix. How many ways one can choose, with repetition allowed, r objects from n distinguishable objects ?

x. **Observation.** With repetition allowed, the number of ways to choose r objects from n distinguishable objects is $\binom{n+r-1}{r}$.

xi. Picking with replacement (picking 5 cards from a deck with replcement) or Picking with repetition (picking 10 icecream cones from 5 flavors).

2. **Pigeonhole (Dirichilet) Principle**

(a) In a group of 367 people, two people must have same birth day.

(b) There are two person in Bhubaneswar and Cuttack having same number of hairs in their body. (Population: 9 millions and # hairs per person $\leq$ 7 millions)

(c) If one collects 10 points from an equilateral triangle of sides having length 1 unit, then there must be 2 points having distance atmost $\frac{1}{3}$ unit.

(d) Pigeonhole principle.

(e) Let acquaintance relation is symmetric (i.e., A knows B iff B knows A). In a group of $50(n > 1)$ people, two people must have same number of acquaintances [2].

(f) One of $101(n+1)$ numbers from the set $\{1, 2, \ldots, 200(2n)\}$ is divisible by another? (Hint: write each number as $2^s a$, where a is odd)

(g) A Chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games in total. There must be a sequence of successive days on which he plays 21 games [2].

(h) **(Erdoš and Szekeres)** Given a sequence of $n^2+1(pq+1)$ distinct integers, either there is an increasing subsequence of $n+1(p+1)$ terms or a decreasing subsequence of $n+1(q+1)$ terms [2, 4].

(i) Ramsey number/theory.

13th January 2010

3. **Principle of Inclusion and Exlusion (PIE)**

(a) In a class of NISER 1st semester, the students failed only in Mathematics and Physics subjects. If the total number of students failed is 7, number of studens failed in Mathematics is 5 and number of students failed in Physics is 3, then how many students failed in both Physics and Mathematics ?

(b) Principle of Inclusion and Exclusion (Using set and using the property satisfied by elements of sets [4]).

(c) How many 4-letter words begin or end with a vowel? ((without)using PIE)

(d) How many integers between 1 and 1000 are (i) not divisible by either 2 or 5 (ii) not divisible by $2, 5$ or 11 ?

(e) Proving Euler's ϕ function [4, Page 410].

(f) Derangements [3, Page 32];
The Hatcheck Problem: How many ways can a hatcheck girl hand back the n hats of n gentlemen, 1 to each gentleman, with no man getting his own hat ?

14th January 2010

4. **Counting Through Recurrence Relation [4, 3]**

(a) Simple/compound interest, Story of Chess inventor, Fibonacci Sequence;

(b) A recurrence relation for a sequence $\{a_i\}, i \geq 0$ or 1 that defines a_n in terms of $a_0, a_1, \ldots, a_{n-1}$ and $n > k$ for some particular integer k, with the terms $a_0, \ldots, a_k$ called initial/boundary condition.

(c) How many regions do n straight lines (non-parallel and no 3 lines intersect at the same point) divide the plane ? (Ans. $a_n = a_{n-1} + (n+1), a_1 = 2$)

(d) Counting the number of decimal strings of length $n+1$ which contain even number of 0s. (Ans: $C_{n+1} = 9C_n + (10^n - C_n) = 8C_n + 10^n, C_1 = 9$)

(e) Counting the number of binary strings of length n which do not contain any consecutive 0s. (Ans: $C_n = C_{n-1} + C_{n-2}, C_1 = 2, C_2 = 3$)

(f) Tower of Hanoi/Brahma.

(g) Counting number of de-arrangements of n objects. (Ans: $D_n = (n-1)(D_{n-1} + D_{n-2})$)

(h) Counting the number of different partitions of $n+1$-element set (i.e., Bell numbers). (Ans: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k$)

21st, 27th, 28th January 2010

5. **Putting r balls into n boxes [4, Page 51], [3, Page 35]**

(a) Distinguishable and In-distinguishable objects.

(b) Number of ways can r-distinguishable balls be put into n-distinguishable boxes. (e.g., Count the number of functions from A to B) (Ans: n^r)

(c) Item 5b with condition that no box is empty (e.g., count the number of onto functions from A to B) (Ans: $\sum_{k=0}^{n}(-1)^k \binom{n}{r}(n-k)^r$)

(d) Number of ways can r-distinguishable balls be put into n-indistinguishable boxes with no box is empty. (This number is denoted by $\left\{ \begin{array}{c} r \\ n \end{array} \right\}$ or $S(r,n)$ and is called as Stirling number of the second kind. Number of ways to partition a set of r things into n non-empty subsets. $B_r = \sum_{i=1}^{r} \left\{ \begin{array}{c} r \\ i \end{array} \right\}$) (Ans: $\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \binom{n}{r}(n-k)^r$ or, $\left\{ \begin{array}{c} r \\ n \end{array} \right\} = \left\{ \begin{array}{c} r-1 \\ n-1 \end{array} \right\} + n \left\{ \begin{array}{c} r-1 \\ n \end{array} \right\}, 1 < n < r, \left\{ \begin{array}{c} r \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} r \\ r \end{array} \right\} = 1$)

(e) Number of ways can r-distinguishable balls be put into n-indistinguishable boxes.(Number of ways to partition a set of r things into at most subsets.) (Ans. $\sum_{i=1}^{n} \left\{ \begin{array}{c} r \\ i \end{array} \right\}$)

(f) Number of ways can r-indistinguishable balls be put into n-distinguishable boxes. (Ans: $\binom{n+r-1}{r}$ Item 1(e)viii)

(g) Item 5f with condition that no box is empty (Ans: $\binom{r-1}{r-n} = \binom{r-1}{n-1}$ Item 1(e)viii)

(h) Number of ways can r-indistinguishable balls be put into n-indistinguishable boxes with no box is empty. (Partitioning an integer $r > 0$ into n parts) (Ans: $\Pi(r,n) = \Pi(r-1,n-1) + \Pi(r-n,n), \Pi(r,n) = 0$ if $n > r$, $\Pi(r,r) = \Pi(r,1) = 1$)

(i) Number of ways can r-indistinguishable balls be put into n-indistinguishable boxes. (Partitioning an integer $r > 0$ into atmost n parts) (Ans: $\Pi(r+n,n)$)

NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, BHUBANESWAR

Assignment I(MA401): Combinatorics and Graph Theory

(Enumerative Combinatorics: PHP, PIE, Recurrence)

Deadline: 04.02.2010 **Total mark: No Counting**

(a) If $2^n + 1$ lattice points are chosen in $\mathcal{R}^n$, prove that there is a pair of these chosen points such that the line segment joining them contains a lattice point in $\mathcal{R}^n$.

(b) A student has 35 days to prepare for a test. From past experience he knows that he will not require more than 60 hours of study. He studies at least one hour per day. Show that no matter how he schedules his study (a whole number of hours per day, however), there exist a succession of t days during which he studies exactly 9 hours.

(c) If $a_1, a_2, ..., a_m$ are integers (not necessarily distinct) then prove that there exist k and l such that $\sum_{i=1}^{l} a_{k+i}$ is divisible by m.

(d) Show that if $m + 1$ integers are chosen from $n + 1, n + 2, ..., n + 2m$ integers, then some pairs of the chosen ones have gcd equal to 1.

(e) Determine the number of permutations of S_n having no fixed points.

(f) In a practice session of penalty shot in soccer, a player has 50% of chance to convert a shot into a goal. What is the probability that the player makes no consecutive goals in n many shots ?

(g) Suppose that 1985 points are given inside a unit cube. Show that one can always choose 32 of them in such a way that every closed polygon with these points as vertices has perimeter less than $8\sqrt{3}$.

(h) Let $a_1, a_2, ..., a_n$ be a permutation of the integers $1, 2, ..., n$ where n is odd. Show that the product $\prod_{i=1}^{n}(a_i - i)$ is an even integer.

(i) Find the number of n-digit codewords from the alphabet $\{0, 1, \ldots, 9\}$ in which the digits $3, 5$ and 7 appear at least once.

(j) In any party of n people, $n \geq 2$, let h_i be the number of people with whom the i^{th} people shook hands with. Show that $\sum_{i=1}^{n} h_i$ is even and that two of h_i are equal.

(k) In statistical mechanics, suppose we have the system of 8 indistinguishable photons and 4 different energy levels of the photons. If we consider that any distribution of photons to energy levels to be equally likely, then find out the probability of the occurence of exactly 2 photons at each energy level (the generalization is called Bose-Einstein statistics).

(l) Each of n gentlemen checks both a hat and an umbrella. Both the hats and umbrellas are returned at random independently. What is the probability that no man gets back both his hat and his umbrella.

(m) Prove that in any party of 6 or more people, there are 3 mutually acquainted people or else 3 mutually unacquainted people.

(n) We are given 81 coins and we know that exactly one of them is counterfeit and it's weight is lower than the others and all others have the same weight. Locate the counterfeit coin by using 4 weighings by a pan balance.

(o) Let $S = (x_1, x_2, ..., x_{n^2+1})$ be a sequence of distinct integers. Prove that S has a decreasing subsequence or an increasing subsequence of length $n + 1$.
(subsequence: if $a_1, \ldots, a_n$ be a sequence then $a_{i_1}, \ldots . a_{i_k}$ is a subsequence provided $1 \leq i_1 < \ldots < i_k \leq n$)

(p) How many n-digit words from the alphabet $\{0, 1, 2\}$ are such that neighbours differ at most by 1 ?

(q) Let $F(n, r)$ = number of n-permutations with exactly r cycles (Stirling number of first kind). Prove the recurrence

$$F(n,r) = F(n-1,r-1)+(n-1)F(n-1,r), \text{ for } 1 < r < n, F(n,1) = (n-1)!, F(n,n) = 1$$

(r) *Given n (distinguishable) bins and m (indistinguishable) balls, how many arrangements are possible such that no bin has greater than r balls?

6. **Generating Function [4, 3]**

(a) Power Series: a power series is an infinite series of the form $\sum_{i=0}^{\infty} a_k x^k$. Radius of Convergence.

(b) Taylor series expansion: Suppose $f(x)$ is a function having darivatives of all orders for all x in an interval containing 0.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

(c) Some useful expansions

i. $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, for $|x| < 1$;

ii. $e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$, for $|x| < \infty$;

iii. $sinx = \sum_{i=0}^{\infty} (-1)^k \frac{1}{(2i+1)!} x^{2i+1}$, for $|x| < \infty$;

iv. $\log(1+x) = \sum_{i=1}^{\infty} (-1)^{k+1} \frac{1}{i} x^i$, for $|x| < 1$.

(d) Generating Function: The generating function for the sequence $(a_k), k \geq 0$ is defined to be

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x^1 + a_2 x^2 + \ldots$$

(e) Some Examples: Find the generating function of

i. $a_k = \binom{n}{k}$;

ii. $a_k = \frac{1}{(k+2)!}$;

(f) Some Examples: Find the sequence of the generating function

i. $f(x) = \frac{x^2}{1-x}$;

ii. $f(x) = cosx$;

iii. $f(x) = \frac{1}{(1+x)^2}$.

(g) Operating on generating functions: Let $A(x), B(x)$ and $C(x)$ be the generating functions for the sequences $(a_k), (b_k)$ and (c_k) respectively. If

i. $A(x) = B(x) + C(x)$, then $a_k = b_k + c_k$;

ii. $A(x) = B(x)C(x)$, then $a_k = \sum_{i=0}^{k} b_i c_{k-i}$. The sequence (a_k) is called convolution of the two sequences (b_k) and (c_k).

iii. Find the sequence of $f(x) = \frac{1}{(1+x)^2}$;

iv. Find the sequence of $f(x) = \frac{1+x+x^2+x^3}{1-x}$.

(h) Examples:

i. How many ways are there to distribute 10 balls to 3 persons ?

ii. How many ways are there to distribute 10 balls to 3 persons such that each person can get at least 2 balls [and at most 5 balls]?

iii. How many ways are there to select 25 balls from unlimited supplies of red, white and blue balls such that at least 3 red balls and at most 5 white balls are selected ?

iv. How many ways one can make 2 dollars using pannies, nickels and dimes ?
v. How many non-negative/positive solutions are there of $x_1 + 7x_2 + 3x_3 = 100$?

4th February 2010

(i) Exponential generating function [4]

i. Exponential generating function for the sequence $(a_k), k \geq 0$ is defined to be

$$H(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 \frac{x^0}{0!} + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \dots$$

ii. $H(x)$ for the sequence $(a_k = 1)$;
iii. $H(x)$ for the sequence $(a_k = \alpha^k)$;
iv. How many words of length at most 5 are there using there letters say, a, b and c (of atmost one a, at most one b and atmost 3 cs).
How many multi-subsets of size at most 5 are there using there letters say, a, b and c (of atmost one a, at most one b and atmost 3 cs).
v. Number of distingushable permutations of length k using p types of objects with upto n_i objects from ith object.
vi. Number of ways to put n distinguishable balls into k distinguishable cells. Compare with the result we found earlier. [4, page 325].

NATIONAL INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, BHUBANESWAR

Quiz I(MA401): Combinatorics and Graph Theory

(Enumerative Combinatorics: PHP, PIE, Recurrence, Balls into Cells)

Date : 16th February, 2010

Max Time: 1 **hrs** **Total Mark:** 40

1. Find a closed formula of the number of regions that are created by n-mutually intersecting (intersecting at two points) circles in a plane such that no-three circles intersect at a point. [5]
2. How many ways (recursion formula) can $n+1$ numbers be multiplied together ? [5]
3. i. How many symmetric functions $f : \{0,1\}^6 \mapsto \{1,2,3\}$ exist ? [5]

   ii. How many rotation symmetric functions $f : \{0,1\}^6 \mapsto \{1,2,3\}$ exist ? [10]

   Note: A function is called symmetric if it outputs the same value for all the inputs of same weight (i.e., it is invariant under all permutations). Similarly, a function is called rotation symmetric if it invariant under cyclic permutations.
4. i. Find the sequence of the generating functions $\frac{1-7x}{1-5x+6x^2}$. [5]

   ii. Find the generating function for the number h_n of the solutions of the equation

   $$e_1 + e_2 + \ldots + e_k = n$$

   in non-negative odd integers $e_1, e_2, \ldots, e_3$. [5]
5. Prove that one of the $k+1$ numbers from $\{1, 2, \ldots, 2k\}$ is divisible by another. [5]

11th, 12th, 17th February 2010

7. **Solving Reccurence Relations [4, 1]**

   (a) Linear homogeneous reccurence relation:

      i. Linear homogeneous reccurence relation: $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_pa_{n-p}$, $n \geq p$, where $c_1, \ldots, c_p$ are constants and $c_p \neq 0$.

      ii. In general, we the initial conditions are disregarded, a recurrence relation has many solutions.

      iii. Theorem [1, 4]: If q is root of the polynomial

      $$x^p - c_1x^{p-1} - c_2x^{p-2} - \cdots - c_p = 0$$

      iff $a_n = q^n$ is a solution of the linear homogeneous recurrence relation

      $$a_n - c_1a_{n-1} - c_2a_{n-2} - \ldots - c_pa_{n-p} = 0, n \geq p, c_p \neq 0$$

      iv. $x^p - c_1x^{p-1} - c_2x^{p-2} - \cdots - c_p = 0$ is called the *characteristic equation* of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_pa_{n-p}$, $n \geq p$, where $c_1, \ldots, c_p$ are constants and $c_p \neq 0$.

      v. Theorem [1, 4]: If the characteristic polynomial equation has p distinct roots $q_1, q_2, \ldots, q_p$, for each initial condition there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that $\lambda_1q_1 + \lambda_2q_2 + \ldots + \lambda_pq_p$ is the solution.

      vi. Solve $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}, n \leq 3$ subject to the initial value $h_0 = 1, h_1 = 2, h_2 = 0$.

      vii. Compute F_k, the kth Fibonnaci number;

      viii. Compute number of ways a string containing '$*'$ (laghu), '$-'$ (guru) and ' $'$ (space) such that no string contains two consecutive ' $'$s (spaces).

      ix. Let $q_1, \ldots, q_t$ be the distinct roots of the of the characteristic equation $x^p - c_1x^{p-1} - c_2x^{p-2} - \cdots - c_p = 0$ with multiplicity $s_1, \ldots, s_t$, then the solutions of the corresponding reccurence solution is

      $$\sum_{i=1}^{t}(\lambda_{i,1} + n\lambda_{i,2} + \ldots + n^{s_i-1}\lambda_{i,s_i})q_i^n$$

      x. Solve the reccurence $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$, $a_1 = 2$.

      xi. Solve the reccurence $a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4}$ with $a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 2$.

   (b) Using Generating Functions

      i. Solve $a_{n+1} = 2a_n$ with $a_0 = \frac{1}{2}$

      ii. Legitmate codewords: How many codewords of length k from the alphabet $\{0, 1, 2, 3\}$ having an even number of 0's are there ? ($a_{k+1} = a_k + 4^k$, $a_1 = 3$)

      iii. Solve Derangement Problem;

      iv. Simultaneous Equations for generating functions [4, Section 6.1.4, 6.3.3]

18th, 22nd, 23rd, 24th February 2010

8. **The Polya Theory of Counting [4]**

   (a) Group action [3];

   (b) Permutation group;

   (c) Equivalence class by the action of permutation group;

(d) Orbit, invariant under a permutation($inv(\pi)$), stabilizer($st(a)$);

(e) Lemma: Suppose that G is a group of permutations on a set A and a is in A. Then $|st(a)|.|C(a)| = |G|$.

(f) Burnside's lemma: Let G be a group of permuatations of a set A and let S be the equivalence relation on A induced by G. Then the number of equivalence classes in S is given by $\frac{1}{|G|}\sum\limits_{\pi \in G} inv(\pi)$.

(g) Equivalent coloring problem;

(h) Cycle decomposition of permutation;

(i) A special case of Polya's theorem: Suppose that G is a permutation group of the set A and $C(A,R)$ is the set of colorings of elements of A using colors in R, a set of m elements. Then the number of didtinct colorings in $C(A,R)$ is given by

$$\frac{1}{|G|}[m^{cyc(\pi_1)} + m^{cyc(\pi_2)} + \ldots + m^{cyc(\pi_k)}]$$

where $G = \{\pi_1, \ldots, \pi_k\}$.

(j) Number of colorings of 2×2 square by 2 colors which are equivalent under rotation by multiples of 90^0.

(k) Number of necklaces of 4 beads by 3 colors which are equivalent under rotations and flips.

(l) Number of rotation symmetric Boolean functions on n variables.

(m) Number of dihedral group invariant Boolean functions on n variables.

(n) Polya's theorem

i. Cycle index of a permutation group;

ii. The inventory of colorings(weight of colors, colorings, inventory of a set of colorings);

iii. If colorings f and g are equivalent, they have the same weight;

iv. Examples of inventory of colorings;

v. Polya's Theorem: Suppose that G is a group of permutations on a set D and $C(D,R)$ is the collection of all colorings of D using colors in R. If w is a weight assignment on R, the pattern inventory of colorings in $C(D,R)$ is given by

$$P_G(\sum_{r \in R} w(r), \sum_{r \in R}[w(r)]^2, \ldots, \sum_{r \in R}[w(r)]^k)$$

where $P_G(x_1, x_2, \ldots, x_k)$ is the cycle index.

vi. Lemma: Suppose that D is divided up into disjoint sets $D_1, D_2, \ldots, D_p$. Let C be the subset of $C(D,R)$ that consists of all colorings f with the property that if $a, b \in D_i$, some i, then $f(a) = f(b)$. Then the inventory of the set C is given by

$$[w(1)^{|D_1|} + w(2)^{|D_1|} + \ldots + w(m)^{|D_1|}] \times [w(1)^{|D_2|} + w(2)^{|D_2|} + \ldots + w(m)^{|D_2|}] \times$$
$$\ldots \times [w(1)^{|D_p|} + w(2)^{|D_p|} + \ldots + w(m)^{|D_p|}].$$

vii. Lemma: Suppose that $G^* = \{\pi_1^*, \pi_2^*, \ldots, \pi_k^*\}$ is a group of permutations of $C(D,R)$. For each $\pi^* \in G^*$, let $\overline{w}(\pi^*)$ be the sum of the weights of all colorings f in $C(D,R)$ left invariant by π^*. Suppose that $C_1, C_2, \ldots, C_t$ are the equivalence classes of colorings and $w(C_i)$ is the common weight of all f in C_i. Then

$$w(C_1) + w(C_2) + \ldots + w(C_t) = \frac{1}{|G^*|}[\overline{w}(\pi_1^*) + \overline{w}(\pi_2^*) + \ldots + \overline{w}(\pi_k^*)].$$

viii. Proof of Polya's theorem.

References

[1] R. A. Brualdi. *Introductory Combinatorics.* Pearson, 2008.

[2] A. Engel. *Problem-Solving Strategies.* Springer, 2003.

[3] G. E. Martin. *Counting: The Art of Enumerative Combinatorics.* Springer, 2001.

[4] F. Roberts and B. Tesman. *Applied Combinatorics.* Prentice Hall, 2004.